

Resonance saturation in the odd-intrinsic parity sector of low-energy QCD

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Abstract

Using the large N_C approximation we have constructed the most general chiral resonance Lagrangian in the odd-intrinsic parity sector that can generate low energy chiral constants up to $\mathcal{O}(p^6)$. Integrating out the resonance fields these $\mathcal{O}(p^6)$ constants are expressed in terms of resonance couplings and masses. The role of η' is discussed and its contribution is explicitly factorized. Using the resonance basis we have also calculated two QCD Green functions of currents: $\langle VVP \rangle$ and $\langle VAS \rangle$ and found, imposing high energy constraints, additional relations for resonance couplings. We have studied several phenomenological implications based on these correlators from which let us mention here our prediction for the π^0 -pole contribution to the muon $g-2$ factor: $a_\mu^{\pi^0} = 65.8(1.2) \times 10^{-11}$.

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1 Introduction

As is well known, there are two regimes where the QCD dynamics of the current correlators is well understood. The first one corresponds to the high energies where the asymptotic freedom allows to use the perturbative approach in terms of the strong coupling constant α_s and where the asymptotics of the correlators for large euclidean momenta is governed by operator product expansion (OPE). The second well understood region is that of low external momenta where the dynamics is constrained by the spontaneously broken chiral symmetry. As a consequence, the dominant contributions to the correlators and related amplitudes of the processes under interest come from the octet of the lightest pseudoscalar mesons (π , K , η) which are the corresponding (pseudo)Goldstone bosons (GB). The correlators can be studied here by means of Chiral Perturbation Theory (ChPT)[1, 2, 3], which is the effective Lagrangian field theory for this region, in terms of systematic simultaneous expansion in powers (and logs) of the momenta and quark masses. The applicability of ChPT extends up to the hadronic scale $\Lambda_H \sim 1\text{GeV}$ which corresponds to the onset of non-Goldstone resonances and where the ChPT expansion fails to converge.

OPE and ChPT provides us with asymptotic behaviour of the correlators in different regimes, however, both these approaches need further non-perturbative long-distance piece of information which is not known from the first principles, namely the values of the vacuum condensates for OPE and the values of the effective low-energy constants (LECs) for ChPT. In the latter case the LECs parameterize our lack of detailed information on the non-perturbative dynamics of the degrees of freedom above the hadronic scale Λ_H and are connected with the order parameters of the spontaneously broken chiral symmetry. The predictivity of ChPT heavily relies on their determination. At the order $O(p^6)$, which corresponds to the recent accuracy of the NNLO ChPT calculation (for a comprehensive review and further references see [4]), 90+4 LECs in the even intrinsic parity sector [5, 6] and 23 LECs in the odd sector¹ [7, 8] appear in the effective Lagrangian. Though only special linear combinations of them are relevant for particular physical amplitudes, the uncertainty in their estimation is usually the weakest point of the interconnection between the theory and experiment.

Dispersion representation of those correlators which are order parameters of the chiral symmetry breaking (and therefore do not get any genuine perturbative contribution) enables to make use of information on the asymptotics both in the low and high energy regions and to relate the unknown LECs to the properties of the corresponding spectral functions in terms of the chiral sum rules [2, 9, 10, 11, 12]. These are usually assumed to be saturated by the low-lying resonant states; such an assumption (known as resonance saturation hypothesis) connects the LECs to the phenomenology of resonances in the intermediate energy region $1\text{GeV} \leq E < 2\text{GeV}$. Though the inclusion of only finite number of resonances has been questioned in the literature [13, 14], it proved to be consistent in the $O(p^4)$ case with other phenomenological determinations of LECs.

The necessary ingredient of the resonance saturation approach to the determination / estimation of LECs is the phenomenological information on the physics of the lowest resonances. It can be conveniently parameterized by means of suitable phenomenological Lagrangian. Along with the chiral symmetry the guiding theoretical principles for its construction are those based on the large N_C expansion of QCD [15]. Within the leading order in $1/N_C$ the correlators of the quark bilinears are given by an infinite sum of contributions of narrow meson resonance states the mass of which scales as $O(N_C^0)$ and the interaction of which is suppressed by an appropriate power of $1/\sqrt{N_C}$. Such a large N_C representation of the correlators can be reconstructed using effective Lagrangian \mathcal{L}_∞ including GB and infinite tower of resonance fields with couplings of the order $O(N_C^{1-n/2})$ according to the number n of the resonance fields in the interaction vertices. The $1/N_C$ expansion is equivalent to the quasi-classical expansion, thus at the leading order only the tree graphs contribute and each additional loop is suppressed by one power of $1/N_C$.

Though \mathcal{L}_∞ is not known from the first principles, the information on the large N_C hierarchy of the individual operators together with general symmetry assumptions allows one to construct all the relevant terms necessary to determine the LECs in the leading order of the large N_C expansion up to given chiral order. The large N_C approximation of LECs can be then formally achieved by means of the integrating out the resonance fields from the Lagrangian \mathcal{L}_∞ . Formally one gets LECs expressed in terms of the (from the first principles unknown) masses and couplings of the infinite tower of resonances.

The large N_C inspired phenomenological Lagrangian suitable for the resonance saturation

¹These numbers of LECs are relevant for $SU(3)$ variant of ChPT. In the $SU(2)$ case we get 53+4 LECs in the intrinsic parity even sector and 5(13) LECs in the odd sector.

program for LECs can be then obtained as an approximation to \mathcal{L}_∞ where only finite number of resonances is kept. Such a truncation of \mathcal{L}_∞ seems to be legitimate at low energies where the contribution of the higher resonances is expected to be suppressed. However, the lack of effective cut-off scale which could play here a role analogous to Λ_H for ChPT prevents us to interpret the resonance phenomenological Lagrangian as a well defined effective theory in the usual sense. It is rather a QCD inspired phenomenological model which should share as much common features with QCD as possible. The latter principle generally puts various constraints on its effective couplings. For instance, the finite number of resonances involved generally corrupts the asymptotic behaviour of the correlators required by perturbative QCD and OPE. However, it is natural to expect that for the correlators which are order parameters of the spontaneous chiral symmetry breaking that the latter behaviour extends down to the region of applicability of the phenomenological Lagrangian and thus it is desirable to ensure the correct asymptotics by means of adjusting its couplings. This is however not enough to fix all of them (often it is even not possible to satisfy all the OPE requirements at once by a finite set of resonances) therefore further phenomenological input is needed.

At the leading order in $1/N_C$, the above strategy for determination of LECs is essentially equivalent to the similar approach known as Minimal Hadronic Ansatz (MHA)[16]. Within this approach the correlators are approximated by meromorphic function with correct pole structure corresponding to the resonance poles and the free parameters are fixed both by OPE constraints and experimental inputs. Only minimal number of resonances is taken into account, just those necessary to satisfy all the relevant OPE (when only the lowest resonances in each channel are included, the method is called Lowest Meson Dominance (LMD) ansatz [16, 12], but in this case not all OPE constraints are guaranteed to be met [12, 17]). Matching this ansatz to the low energy ChPT expansion enables to determine relevant linear combinations of LECs.

The method based on the resonance Lagrangian is however little bit more general than MHA or LMD. On one hand it enables to determine (at least in principle) the individual LECs, not only their linear combinations connected with particular correlators, on the other hand it provides a natural framework for going beyond the leading order in $1/N_C$ by means of integrating out the resonances at one loop level [18, 19, 20, 21, 22] which also takes correctly into account the renormalization scale dependence of the LECs.

The above principles of construction of phenomenological Lagrangian with resonances are known since 1989 when the seminal paper [23] on what is now known as Resonance Chiral Theory (R χ T) was published. In this paper the resonance saturation of the $O(p^4)$ LECs has been studied systematically while the $O(p^6)$ LECs of the even intrinsic parity sector of ChPT has been systematically analyzed 17 years later in [24]. For a recent review and further references see [25].

The study of the odd intrinsic parity sector of R χ T with vector resonances and corresponding saturation of the LECs for the $O(p^6)$ anomaly sector of ChPT started in [26] and [27, 12, 28], where also axial vector resonances has been included and where the particular operator basis of the R χ T Lagrangian contributing to the correlators under interest has been constructed. The influence of pseudoscalar resonances on the odd intrinsic parity LECs has been studied in [11, 12] and corresponding part of R χ T Lagrangian has been constructed in [29] (see also [30]). In this paper we resume this effort and construct the most general odd intrinsic parity sector of the R χ T Lagrangian including the lowest multiplets of the vector $V(1^{--})$, axial-vector $A(1^{++})$, scalar $S(0^{++})$ and pseudoscalar $P(0^{-+})$ resonances. In the 0^{-+} channel we introduce thus beside the GB also the lowest non-GB resonance multiplet

and therefore we go beyond the LMD approximation (our correlators then correspond to what is called in [12] as LMD+P ansatz). The resulting Lagrangian is then used for the lowest resonance saturation of the $O(p^6)$ anomaly sector of ChPT. We also illustrate the general strategy of matching the correlators with OPE on the concrete example of $\langle VVP \rangle$ and $\langle VAS \rangle$ three point functions and discuss related phenomenological applications.

The paper is organized as follows. In Sect. 2 we fix our notation and remind briefly the principles of the construction of the Lagrangian of the $R\chi T$. Sec. 3 is devoted to the presentation of the complete basis of the odd intrinsic parity sector of $R\chi T$. In Sect. 4 we discuss related phenomenological applications and in Sect. 5 we give the result of the resonance saturation of the odd intrinsic parity $O(p^6)$ LECs. A brief summary is given in Sec 6. The large N_C counting of the relevant operators is discussed in Appendix A and the operator redefinitions and reduction of the Lagrangian is studied in Appendix B.

2 The Resonance Chiral Theory

In what follows we will work in the chiral limit. The standard basic building block which includes the octet of GB (here we assume that η' has been already integrated out from our effective Lagrangian, for details see Appendix A) is

$$u(\phi) = \exp \left(i \frac{\phi}{\sqrt{2}F} \right), \quad (1)$$

where $\phi = \frac{1}{\sqrt{2}}\lambda^a \phi^a$, λ^i being a standard Gell-Mann matrix and

$$\phi(x) = \begin{pmatrix} \frac{1}{\sqrt{2}}\pi^0 + \frac{1}{\sqrt{6}}\eta_8 & \pi^+ & K^+ \\ \pi^- & -\frac{1}{\sqrt{2}}\pi^0 + \frac{1}{\sqrt{6}}\eta_8 & K^0 \\ K^- & \bar{K}^0 & -\frac{2}{\sqrt{6}}\eta_8 \end{pmatrix}. \quad (2)$$

One can form the basic covariant tensors [31], [5]

$$\begin{aligned} u_\mu &= u_\mu^\dagger = i \{ u^\dagger (\partial_\mu - i r_\mu) u - u (\partial_\mu - i \ell_\mu) u^\dagger \}, \\ \chi_\pm &= u^\dagger \chi u^\dagger \pm u \chi^\dagger u, \\ f_\pm^{\mu\nu} &= u F_L^{\mu\nu} u^\dagger \pm u^\dagger F_R^{\mu\nu} u, \\ h_{\mu\nu} &= \nabla_\mu u_\nu + \nabla_\nu u_\mu, \end{aligned} \quad (3)$$

with $\chi = 2B_0(s + ip)$, where s and p stand for the scalar and pseudo-scalar external sources. Vector source v^μ and axial-vector source a^μ are then related to the right and left sources r^μ and ℓ^μ by relations $v^\mu = \frac{1}{2}(r^\mu + \ell^\mu)$ and $a^\mu = \frac{1}{2}(r^\mu - \ell^\mu)$ respectively, and $F_{L,R}^{\mu\nu}$ the corresponding left and right field-strength tensors:

$$F_R^{\mu\nu} = \partial^\mu r^\nu - \partial^\nu r^\mu - i[r^\mu, r^\nu], \quad F_L^{\mu\nu} = \partial^\mu \ell^\nu - \partial^\nu \ell^\mu - i[\ell^\mu, \ell^\nu]. \quad (4)$$

The covariant derivative is defined by

$$\nabla_\mu X = \partial_\mu X + [\Gamma_\mu, X], \quad (5)$$

where the chiral connection is

$$\Gamma_\mu = \frac{1}{2} \{ u^\dagger (\partial_\mu - i r_\mu) u + u (\partial_\mu - i \ell_\mu) u^\dagger \}. \quad (6)$$

Inspired by the large N_C limit the GB couple to massive $U(3)$ multiplets of the type $V(1^{--})$, $A(1^{++})$, $S(0^{++})$ and $P(0^{-+})$, denoted generically as a nonet field R . This field can be decomposed into octet R_8 and singlet R_0 via

$$R = \frac{1}{\sqrt{3}}R_0 + \sum_i \frac{\lambda_i}{\sqrt{2}}R_i. \quad (7)$$

The explicit form of the vector multiplet $V(1^{--})$ is

$$V_{\mu\nu} = \begin{pmatrix} \frac{1}{\sqrt{2}}\rho^0 + \frac{1}{\sqrt{6}}\omega_8 + \frac{1}{\sqrt{3}}\omega_1 & \rho^+ & K^{*+} \\ \rho^- & -\frac{1}{\sqrt{2}}\rho^0 + \frac{1}{\sqrt{6}}\omega_8 + \frac{1}{\sqrt{3}}\omega_1 & K^{*0} \\ K^{*-} & \bar{K}^{*0} & -\frac{2}{\sqrt{6}}\omega_8 + \frac{1}{\sqrt{3}}\omega_1 \end{pmatrix}_{\mu\nu}, \quad (8)$$

(and similarly for other types). We use here the antisymmetric tensor field for description of the spin-1 resonances. The reason is that though it is in principle equivalent to the Proca field formalism (see [32] and [33] for the general discussion of the equivalence at the order $O(p^4)$ and $O(p^6)$ respectively and [34] for particular discussion of the one-loop equivalence), the antisymmetric tensor field naturally couples to the lowest order $O(p^2)$ chiral building blocks without derivatives and therefore it does not require additional contact terms necessary to compensate the wrong high energy behaviour of amplitudes and form factors under interest. Moreover, when using the Proca field without such contact terms it is not possible to saturate the $O(p^4)$ LECs in the even intrinsic parity sector and for the similar reason also the LECs in the $O(p^6)$ odd intrinsic parity sector.

According to the large N_C counting of interaction vertices with resonances we can organize the Lagrangian $\mathcal{L}_{R\chi T}$ of $R\chi T$ as an expansion in the number of resonance fields,

$$\mathcal{L}_{R\chi T} = \mathcal{L}_{GB} + \mathcal{L}_{RR,\text{kin}} + \mathcal{L}_R + \mathcal{L}_{RR'} + \mathcal{L}_{RR'R''} + \dots \quad (9)$$

Here \mathcal{L}_{GB} contains only Goldstone bosons and external sources and includes terms with the same structure as the usual $SU(3)_L \times SU(3)_R$ ChPT Lagrangian, but the coupling constants are generally different. The resonance kinetic terms $\mathcal{L}_{RR,\text{kin}}$, which are of the order $O(N_C^0)$, have the form

$$\mathcal{L}_{RR,\text{kin}} = -\frac{1}{2}\langle \nabla^\mu R_{\mu\nu} \nabla_\alpha R^{\alpha\nu} \rangle + \frac{1}{4}M_R^2 \langle R_{\mu\nu} R^{\mu\nu} \rangle + \frac{1}{2}\langle \nabla^\alpha R' \nabla_\alpha R' \rangle - \frac{1}{2}M_{R'}^2 \langle R' R' \rangle, \quad (10)$$

where R stands for $V^{\mu\nu}$ and $A^{\mu\nu}$ while R' stands for S and P . The terms \mathcal{L}_R , $\mathcal{L}_{RR'}$ and $\mathcal{L}_{RR'R''}$ collect the interaction vertices linear, quadratic and cubic in the resonance fields, respectively.

There is also another type of expansion for $\mathcal{L}_{R\chi T}$. It is based on the ordering according to the contribution to chiral coupling constants. Within this counting, the resonance fields are effectively of the order

$$R = O(p^2), \quad (11)$$

while the chiral building blocks with GB only are counted in a usual way. For \mathcal{L}_{GB} it is therefore just the usual chiral power counting. Combining this with the large N_C expansion (9) we can write

$$\mathcal{L}_{R\chi T} = \mathcal{L}_{GB}^{(2)} + \mathcal{L}_{GB}^{(4)} + \mathcal{L}_{RR,\text{kin}}^{(4)} + \mathcal{L}_{RR,\text{kin}}^{(6)} + \mathcal{L}_R^{(4)} + \mathcal{L}_{GB}^{(6)} + \mathcal{L}_R^{(6)} + \mathcal{L}_{RR'}^{(6)} + \mathcal{L}_{RR'R''}^{(6)} \dots, \quad (12)$$

where the subscript $^{(n)}$ stands for the contribution to $\mathcal{O}(p^n)$ chiral constant. For our further discussion we will explicitly need

$$\mathcal{L}_{GB}^{(2)} = \frac{F^2}{4} \langle u_\mu u^\mu + \chi_+ \rangle. \quad (13)$$

The leading order of the odd intrinsic parity sector of $\mathcal{L}_{GB}^{(4)}$ coincides with the Wess-Zumino-Witten Lagrangian [35] $\mathcal{L}_{WZW}^{(4)}$. For the explicit form of even parity part $\mathcal{L}_{GB}^{(4)}$ and complete $\mathcal{L}_{GB}^{(6)}$ see [3, 5, 8], (see also [7]).

The most general interaction Lagrangian $\mathcal{L}_R^{(4)}$ which is relevant for the saturation of the $O(p^4)$ LECs [23] is linear in resonance fields, namely

$$\begin{aligned} \mathcal{L}_R^{(4)} = & c_d \langle S u^\mu u_\mu \rangle + c_m \langle S \chi_+ \rangle + i d_m \langle P \chi_- \rangle + i \frac{d_{m0}}{N_F} \langle P \rangle \langle \chi_- \rangle + \\ & + \frac{F_V}{2\sqrt{2}} \langle V_{\mu\nu} f_+^{\mu\nu} \rangle + \frac{i G_V}{2\sqrt{2}} \langle V_{\mu\nu} [u^\mu, u^\nu] \rangle + \frac{F_A}{2\sqrt{2}} \langle A_{\mu\nu} f_-^{\mu\nu} \rangle \end{aligned} \quad (14)$$

and all the couplings are of the order $O(N_C^{1/2})$. This is true also for the last term of the first line with two traces which is enhanced due to the η' exchange (see Appendix A, esp. (112)). This term with d_{m0} (depending solely on the singlet component of P) has not yet been studied in the phenomenology as it always contributes to the saturation of LECs together with the large N_C enhanced η' exchange. The complete operator basis of the $O(p^6)$ even intrinsic parity of $R\chi T$ has been constructed in [24].

Integrating out the resonance fields at the tree level we reconstruct the Lagrangian $\mathcal{L}_{\chi PT}$ of ChPT, schematically

$$\exp \left(i \int d^4 x \mathcal{L}_{\chi PT} \right) = \int \mathcal{D}R \exp \left(i \int d^4 x \mathcal{L}_{R\chi T} \right). \quad (15)$$

Effectively up to the order $O(p^6)$ the integration over R is equivalent to the insertion of the solution $R^{(2)}$ of the lowest order equation of motion (i.e. those derived from $\mathcal{L}_{RR,kin}^{(4)} + \mathcal{L}_R^{(4)}$) for resonance field R into the Lagrangian $\mathcal{L}_{R\chi T}$. Because the resonance fields R couples to the $O(p^2)$ building blocks in $\mathcal{L}_R^{(4)}$ and the resonance masses are counted as $O(p^0)$, we are consistent with the chiral counting (11). Finally we get

$$\mathcal{L}_{\chi PT} = \mathcal{L}_\chi^{(2)} + \mathcal{L}_\chi^{(4)} + \mathcal{L}_\chi^{(6)} + \dots$$

with explicit separate contribution from Goldstone bosons part of the $R\chi T$ Lagrangian \mathcal{L}_{GB} and the leading N_C contribution of the resonances

$$\mathcal{L}_\chi^{(2)} = \mathcal{L}_{GB}^{(2)} \quad (16)$$

$$\mathcal{L}_\chi^{(n>2)} = \mathcal{L}_{GB}^{(n)} + \mathcal{L}_{\chi,R}^{(n)}, \quad (17)$$

where particularly

$$\begin{aligned} \mathcal{L}_{\chi,R}^{(4)} &= \left(\mathcal{L}_{RR,kin}^{(4)} + \mathcal{L}_R^{(4)} \right) \Big|_{R \rightarrow R^{(2)}} \\ \mathcal{L}_{\chi,R}^{(6)} &= \left(\mathcal{L}_{RR,kin}^{(6)} + \mathcal{L}_R^{(6)} + \mathcal{L}_{RR'}^{(6)} + \mathcal{L}_{RR'R''}^{(6)} \right) \Big|_{R \rightarrow R^{(2)}}. \end{aligned}$$

The structure of Lagrangians $\mathcal{L}_{GB}^{(n)}$ and $\mathcal{L}_{\chi,R}^{(n)}$ are identical to $\mathcal{L}_{\chi}^{(n)}$, just the couplings are different. Then for generic chiral coupling constants k_{χ} of $\mathcal{L}_{\chi PT}$ we may write

$$k_{\chi} = k_{GB} + k_{\chi,R}, \quad (18)$$

where $k_{\chi,R}$ corresponds to the resonance contribution. The usual hypothesis of resonance saturation assumes k_{GB} to be very small and the resonance contribution $k_{\chi,R}$ is expected to be dominant.

The above resonance saturation strategy and the construction of all relevant operators were studied already in the past. In [23] was found the basis for all relevant resonance operators contributing to $\mathcal{O}(p^4)$ and their contribution to LECs while in [24] the authors presented the extension to $\mathcal{O}(p^6)$ in even-intrinsic-parity sector.

In this paper, we complete this effort presenting the construction of basis and resonance saturation at $\mathcal{O}(p^6)$ in the odd-intrinsic parity sector.

3 Lagrangian of $R\chi T$ in odd-intrinsic parity sector

Before starting the construction of resonance monomials let us summarize the structure of the pure Goldstone-boson part of the odd-intrinsic sector. The leading order starts at $\mathcal{O}(p^4)$ and the parameters are set entirely by the chiral anomaly. The Lagrangian is given by [35] (see also [8]; note we have the same convention for the Levi-Civita symbol, i.e. $\epsilon_{0123} = 1$):

$$\mathcal{L}_{WZW} = \frac{N_C}{48\pi^2} \epsilon^{\mu\nu\alpha\beta} \left\{ \int_0^1 d\xi \langle \sigma_{\mu}^{\xi} \sigma_{\nu}^{\xi} \sigma_{\alpha}^{\xi} \sigma_{\beta}^{\xi} \frac{\phi}{F} \rangle - i \langle W_{\mu\nu\alpha\beta}(U, l, r) - W_{\mu\nu\alpha\beta}(1, l, r) \rangle \right\}, \quad (19)$$

with

$$W_{\mu\nu\alpha\beta} = L_{\mu} L_{\nu} L_{\alpha} R_{\beta} + \frac{1}{4} L_{\mu} R_{\nu} L_{\alpha} R_{\beta} + i L_{\mu\nu} L_{\alpha} R_{\beta} + i R_{\mu\nu} L_{\alpha} R_{\beta} - i \sigma_{\mu} L_{\nu} R_{\alpha} L_{\beta} + \sigma_{\mu} R_{\nu\alpha} L_{\beta} - \sigma_{\mu} \sigma_{\nu} R_{\alpha} L_{\beta} + \sigma_{\mu} L_{\nu} L_{\alpha\beta} + \sigma_{\mu} L_{\nu\alpha} L_{\beta} - i \sigma_{\mu} L_{\nu} L_{\alpha} L_{\beta} + \frac{1}{2} \sigma_{\mu} L_{\nu} \sigma_{\alpha} L_{\beta} - i \sigma_{\mu} \sigma_{\nu} \sigma_{\alpha} L_{\beta} - (L \leftrightarrow R),$$

where we have defined

$$L_{\mu} = u l_{\mu} u^{\dagger}, \quad L_{\mu\nu} = u \partial_{\mu} l_{\nu} u^{\dagger}, \quad R_{\mu} = u^{\dagger} r_{\mu} u, \quad R_{\mu\nu} = u \partial_{\mu} r_{\nu} u^{\dagger}, \quad \sigma_{\mu} = \{u^{\dagger}, \partial_{\mu} u\}$$

and $(L \leftrightarrow R)$ stands also for $\sigma \leftrightarrow \sigma^{\dagger}$ interchange. The power ξ indicates a change of u to $u^{\xi} = \exp(i\xi\phi/(F\sqrt{2}))$. Concerning the $\mathcal{O}(p^6)$ part we will stick to the form introduced in [8]. Let us only note that we will drop the index r and the explicit dependence on the renormalization scale μ from the corresponding LECs C_i^W , but one should have in mind that any C_i^W studied in this text is a renormalized LEC with the scale set to some reasonable value ($\sim M_{\rho}, M_{\eta'}$) to make good sense of the following study.

For the construction of the operator basis in the odd intrinsic parity sector of $R\chi T$ we use the same tools as in [24], where the reader can find further details. First we construct all possible operators built from chiral building blocks and resonance fields that are invariant under all symmetries. Then in order to find the independent basis we use

1. Partial integration
2. Equation of motion

$$\nabla^{\mu} u_{\mu} = \frac{i}{2} \left(\chi - \frac{1}{N_F} \langle \chi - \rangle \right) \quad (20)$$

3. Bianchi identities

$$\nabla_\mu \Gamma_{\nu\rho} + \nabla_\nu \Gamma_{\rho\mu} + \nabla_\rho \Gamma_{\mu\nu} = 0 \quad \text{for} \quad \Gamma_{\mu\nu} = \frac{1}{4}[u_\mu, u_\nu] - \frac{i}{2}f_{+\mu\nu} \quad (21)$$

4. Shouten identity [36]

$$g_{\sigma\rho}\epsilon_{\alpha\beta\mu\nu} + g_{\sigma\alpha}\epsilon_{\beta\mu\nu\rho} + g_{\sigma\beta}\epsilon_{\mu\nu\rho\alpha} + g_{\sigma\mu}\epsilon_{\nu\rho\alpha\beta} + g_{\sigma\nu}\epsilon_{\rho\alpha\beta\mu} = 0 \quad (22)$$

5. Identity

$$\nabla^\mu h_{\mu\nu} = \nabla_\nu h_\mu^\mu - 2[u^\mu, i\Gamma_{\mu\nu}] - \nabla^\mu f_{-\mu\nu} \quad (23)$$

All relevant operators in odd parity sector can be written in the form

$$\mathcal{O}_i^X = \varepsilon^{\mu\nu\alpha\beta} \widehat{\mathcal{O}}_{i\mu\nu\alpha\beta}^X, \quad (24)$$

with the basis for individual monomials $\widehat{\mathcal{O}}_{i\mu\nu\alpha\beta}^X$, with $X = V, A, P, S, VV, AA, SA, SV, VA, PA, PV, VVP, VAS, AAP$; so Lagrangian becomes:

$$\mathcal{L}_{R\chi T}^{(6, \text{odd})} = \sum_X \sum_i \kappa_i^X \mathcal{O}_i^X. \quad (25)$$

The basis of the operators $\widehat{\mathcal{O}}_{i\mu\nu\alpha\beta}^X$ is summarized in Tables 1-7. We have included there only the operators relevant in the leading order in the $1/N_C$ expansion *i.e.* operators with one flavour trace and those with two traces that are enhanced by η' exchange (see Appendix A for details). This represents main result of our work.

As is shown in [24, 33], we can further modify the resonance Lagrangian (25). The reason is that the resonance fields play merely the role of the integration variables in the path integral (15) and can be therefore freely redefined without changing the physical content of the theory. As a consequence we can choose appropriate field redefinition in order to eliminate some subset $\{\mathcal{O}_i^X\}_{(X,i) \in M}$ of $O(p^6)$ operators from $\mathcal{L}_{R\chi T}^{(6, \text{odd})}$ and shift their influence effectively to the $O(p^6)$ terms including the remaining operators $\{\mathcal{O}_i^X\}_{(X,i) \notin M}$ and also to the higher chiral order terms $\mathcal{L}_{R\chi T}^{(>6, \text{odd})}$, symbolically

$$\mathcal{L}_{R\chi T}^{(6, \text{odd})} = \sum_{(X,i)} \kappa_i^X \mathcal{O}_i^X \rightarrow \sum_{(X,i) \notin M} \overline{\kappa}_i^X \mathcal{O}_i^X + \mathcal{L}_{R\chi T}^{(>6, \text{odd})} \quad (26)$$

The possible new terms $\mathcal{L}_{R\chi T}^{(>6, \text{odd})}$ of the order $O(p^8)$ and higher generated by such a redefinition can be neglected because they do not contribute to the $O(p^6)$ LECs when the resonance fields are integrated out. Note however, that after such a truncation we get new Lagrangian

$$\overline{\mathcal{L}}_{R\chi T}^{(6, \text{odd})} = \sum_{(X,i) \notin M} \overline{\kappa}_i^X \mathcal{O}_i^X \quad (27)$$

which is not equivalent with the previous one on the resonance level. On the other hand, the LECs obtained from $\overline{\mathcal{L}}_{R\chi T}^{(6, \text{odd})}$ coincide with those derived from $\mathcal{L}_{R\chi T}^{(6, \text{odd})}$.

The stars in the Tables 1-7 indicate those operators which can be eliminated by the resonance fields redefinitions discussed above and means therefore a redundancy of a given

i	$\hat{\mathcal{O}}_{i\mu\nu\alpha\beta}^V$	i	$\hat{\mathcal{O}}_{i\mu\nu\alpha\beta}^V$
1	$i\langle V^{\mu\nu}(h^{\alpha\sigma}u_\sigma u^\beta - u^\beta u_\sigma h^{\alpha\sigma})\rangle$	11	$\langle V^{\mu\nu}\{f_+^{\alpha\rho}, f_-^{\beta\sigma}\}\rangle g_{\rho\sigma}$
2	$i\langle V^{\mu\nu}(u_\sigma h^{\alpha\sigma} u^\beta - u^\beta h^{\alpha\sigma} u_\sigma)\rangle$	12	$\langle V^{\mu\nu}\{f_+^{\alpha\rho}, h^{\beta\sigma}\}\rangle g_{\rho\sigma}$
3	$i\langle V^{\mu\nu}(u_\sigma u^\beta h^{\alpha\sigma} - h^{\alpha\sigma} u^\beta u_\sigma)\rangle$	13	$i\langle V^{\mu\nu} f_+^{\alpha\beta}\rangle\langle\chi_-\rangle$
4	$i\langle[V^{\mu\nu}, \nabla^\alpha\chi_+]u^\beta\rangle$	14	$i\langle V^{\mu\nu}\{f_+^{\alpha\beta}, \chi_-\}\rangle$
5	$i\langle V^{\mu\nu}[f_-^{\alpha\beta}, u_\sigma u^\sigma]\rangle$	15	$i\langle V^{\mu\nu}[f_-^{\alpha\beta}, \chi_+]\rangle$
6	$i\langle V^{\mu\nu}(f_-^{\alpha\sigma} u^\beta u_\sigma - u_\sigma u^\beta f_-^{\alpha\sigma})\rangle$	16	$\langle V^{\mu\nu}\{\nabla^\alpha f_+^{\beta\sigma}, u_\sigma\}\rangle$
7	$i\langle V^{\mu\nu}(u_\sigma f_-^{\alpha\sigma} u^\beta - u^\beta f_-^{\alpha\sigma} u_\sigma)\rangle$	17	$\langle V^{\mu\nu}\{\nabla_\sigma f_+^{\alpha\sigma}, u^\beta\}\rangle$
8	$i\langle V^{\mu\nu}(f_-^{\alpha\sigma} u_\sigma u^\beta - u^\beta u_\sigma f_-^{\alpha\sigma})\rangle$	18	$\langle V^{\mu\nu} u^\alpha u^\beta\rangle\langle\chi_-\rangle$
9	$\langle V^{\mu\nu}\{\chi_-, u^\alpha u^\beta\}\rangle$		
10	$\langle V^{\mu\nu} u^\alpha \chi_- u^\beta\rangle$		

Table 1: Monomials with one vector resonance field.

i	$\hat{\mathcal{O}}_{i\mu\nu\alpha\beta}^A$	i	$\hat{\mathcal{O}}_{i\mu\nu\alpha\beta}^A$
1	$\langle A^{\mu\nu}[u^\alpha u^\beta, u_\sigma u^\sigma]\rangle$	10	$i\langle A^{\mu\nu} u^\alpha\rangle\langle\nabla^\beta\chi_-\rangle$
2	$\langle A^{\mu\nu}[u^\alpha u^\sigma u^\beta, u^\sigma]\rangle$	11	$i\langle A^{\mu\nu}\{f_-^{\alpha\beta}, \chi_-\}\rangle$
3	$\langle A^{\mu\nu}\{\nabla^\alpha h^{\beta\sigma}, u_\sigma\}\rangle$	12	$i\langle A^{\mu\nu}\{\nabla^\alpha\chi_-, u^\beta\}\rangle$
4	$i\langle A^{\mu\nu}[f_+^{\alpha\beta}, u^\sigma u_\sigma]\rangle$	13	$\langle A^{\mu\nu}[\chi_+, u^\alpha u^\beta]\rangle$
5	$i\langle A^{\mu\nu}(f_+^{\alpha\sigma} u_\sigma u^\beta - u^\beta u_\sigma f_+^{\alpha\sigma})\rangle$	14	$i\langle A^{\mu\nu}[f_+^{\alpha\beta}, \chi_+]\rangle$
6	$i\langle A^{\mu\nu}(f_+^{\alpha\sigma} u^\beta u_\sigma - u_\sigma u^\beta f_+^{\alpha\sigma})\rangle$	15	$\langle A^{\mu\nu}\{\nabla^\alpha f_-^{\beta\sigma}, u_\sigma\}\rangle$
7	$i\langle A^{\mu\nu}(u_\sigma f_+^{\alpha\sigma} u^\beta - u^\beta f_+^{\alpha\sigma} u_\sigma)\rangle$	16	$\langle A^{\mu\nu}\{\nabla_\sigma f_-^{\alpha\sigma}, u^\beta\}\rangle$
8	$\langle A^{\mu\nu}\{f_-^{\alpha\sigma}, h^{\beta\sigma}\}\rangle$		
9	$i\langle A^{\mu\nu} f_-^{\alpha\beta}\rangle\langle\chi_-\rangle$		

Table 2: Monomials with one axial-vector resonance field.

monomial as far as its contribution to the resonance saturation is concerned. The details are shown in Appendix B. Note, however, that this redundancy concerns only the saturation and not the direct calculation of the correlators and amplitudes with resonances in the initial and final states. We will return to this point later on.

In the following section we will demonstrate the use of the resonance basis for two classes of examples. The resonance saturation will be studied in Section 5.

4 Applications

In this section we illustrate applications of the Lagrangian $\mathcal{L}_{R\chi T}^{(6, \text{odd})}$ using two examples. We study two three-point correlators, namely $\langle VVP \rangle$ and $\langle VAS \rangle$, and use both OPE constraints as well as phenomenological inputs to fix the relevant coupling constants. In the first case we also discuss some phenomenological applications in more detail.

i	$\widehat{\mathcal{O}}_{i\mu\nu\alpha\beta}^P$	i	$\widehat{\mathcal{O}}_{i\mu\nu\alpha\beta}^P$	i	$\widehat{\mathcal{O}}_{i\mu\nu\alpha\beta}^S$
1	$\langle P\{f_-^{\mu\nu}, f_-^{\alpha\beta}\} \rangle$	4	$\langle Pu^\mu u^\nu u^\alpha u^\beta \rangle$	1	$\langle S[f_-^{\alpha\beta}, u^\mu u^\nu] \rangle$
2	$i\langle Pu^\alpha f_+^{\mu\nu} u^\beta \rangle$	5	$\langle P\{f_+^{\mu\nu}, f_+^{\alpha\beta}\} \rangle$	2	$i\langle S[f_+^{\mu\nu}, f_-^{\alpha\beta}] \rangle$
3	$i\langle P\{f_+^{\mu\nu}, u^\alpha u^\beta\} \rangle$	6			

Table 3: Monomials with scalar or pseudo-scalar resonance field.

i	Operator $\widehat{\mathcal{O}}_{i\mu\nu\alpha\beta}^{RR}$, $R = V, A$	Operator $\widehat{\mathcal{O}}_{i\mu\nu\alpha\beta}^{RR}$, $R = P, S$
1*	$i\langle R^{\mu\nu} R^{\alpha\beta} \rangle \langle \chi_- \rangle$	
2*	$i\langle \{R^{\mu\nu}, R^{\alpha\beta}\} \chi_- \rangle$	
3	$\langle \{\nabla_\sigma R^{\mu\nu}, R^{\alpha\sigma}\} u^\beta \rangle$	
4	$\langle \{\nabla^\beta R^{\mu\nu}, R^{\alpha\sigma}\} u_\sigma \rangle$	

Table 4: Monomials with two resonance fields of the same kind.

i	Operator $\widehat{\mathcal{O}}_{i\mu\nu\alpha\beta}^{SA}$	i	Operator $\widehat{\mathcal{O}}_{i\mu\nu\alpha\beta}^{SV}$	Operator $\widehat{\mathcal{O}}_{i\mu\nu\alpha\beta}^{SP}$
1*	$i\langle [A^{\mu\nu}, S] f_+^{\alpha\beta} \rangle$	1*	$i\langle [V^{\mu\nu}, S] f_-^{\alpha\beta} \rangle$	
2*	$\langle A^{\mu\nu} [S, u^\alpha u^\beta] \rangle$	2	$i\langle [V^{\mu\nu}, \nabla^\alpha S] u^\beta \rangle$	

Table 5: Monomials with two resonance fields of different kinds.

i	Operator $\widehat{\mathcal{O}}_{i\mu\nu\alpha\beta}^{VA}$	i	Operator $\widehat{\mathcal{O}}_{i\mu\nu\alpha\beta}^{PA}$	i	Operator $\widehat{\mathcal{O}}_{i\mu\nu\alpha\beta}^{PV}$
1*	$i\langle V^{\mu\nu} [A^{\alpha\beta}, u^\sigma u_\sigma] \rangle$	1*	$\langle \{A^{\mu\nu}, P\} f_-^{\alpha\beta} \rangle$	1*	$i\langle \{V^{\mu\nu}, P\} u^\alpha u^\beta \rangle$
2*	$i\langle V^{\mu\nu} (A^{\alpha\sigma} u_\sigma u^\beta - u^\beta u_\sigma A^{\alpha\sigma}) \rangle$	2	$\langle \{A^{\mu\nu}, \nabla^\alpha P\} u^\beta \rangle$	2*	$i\langle V^{\mu\nu} u^\alpha P u^\beta \rangle$
3*	$i\langle V^{\mu\nu} (A^{\alpha\sigma} u^\beta u_\sigma - u_\sigma u^\beta A^{\alpha\sigma}) \rangle$			3*	$\langle \{V^{\mu\nu}, P\} f_+^{\alpha\beta} \rangle$
4*	$i\langle V^{\mu\nu} (u_\sigma A^{\alpha\sigma} u^\beta - u^\beta A^{\alpha\sigma} u_\sigma) \rangle$				
5*	$\langle \{V^{\mu\nu}, A^{\alpha\rho}\} f_+^{\beta\sigma} \rangle g_{\rho\sigma}$				
6*	$i\langle [V^{\mu\nu}, A^{\alpha\beta}] \chi_+ \rangle$				

Table 6: Monomials with two resonance fields of different kinds.

i	Operator $\widehat{\mathcal{O}}_{i\mu\nu\alpha\beta}^{RRR}$
1*	$\langle V^{\mu\nu} V^{\alpha\beta} P \rangle$
2*	$i\langle [V^{\mu\nu}, A^{\alpha\beta}] S \rangle$
3*	$\langle A^{\mu\nu} A^{\alpha\beta} P \rangle$

Table 7: Monomials with three resonance fields.

4.1 VVP Green function revisited

The standard definition of this correlator is

$$\Pi_{\mu\nu}^{abc}(p, q) = \int d^4x d^4y e^{ip \cdot x + iq \cdot y} \langle 0 | T [V_\mu^a(x) V_\nu^b(y) P^c(0) | 0 \rangle \quad (28)$$

with the vector current and the pseudoscalar density defined by

$$V_\mu^a(x) = \bar{q}(x) \gamma_\mu \frac{\lambda^a}{2} q(x), \quad P_\mu^a(x) = \bar{q}(x) i \gamma_5 \frac{\lambda^a}{2} q(x), \quad (29)$$

(our convention is $\gamma_5 = i \gamma^0 \gamma^1 \gamma^2 \gamma^3$). This correlator was already studied in the past, see e.g. [11], [12], [28]. Here we provide a complete result based on our $\mathcal{L}_{\text{R}\chi\text{T}}$, i.e. also with two- and three-resonance vertices that were not considered in [28]. Using Ward identities and Lorentz and parity invariance one can define

$$\Pi(p)_{\mu\nu}^{abc} = d^{abc} \epsilon_{\mu\nu\alpha\beta} p^\alpha q^\beta \Pi(p^2, q^2; r^2). \quad (30)$$

The OPE constraints dictate for high values of all independent momenta

$$\Pi((\lambda p)^2, (\lambda q)^2; (\lambda r)^2) = \frac{B_0 F^2}{2\lambda^4} \frac{p^2 + q^2 + r^2}{p^2 q^2 r^2} + \mathcal{O}\left(\frac{1}{\lambda^6}\right), \quad (31)$$

whereas in the case when only two operators are close to each other one gets

$$\Pi((\lambda p)^2, (q - \lambda p)^2; q^2) = \frac{B_0 F^2}{\lambda^2} \frac{1}{p^2 q^2} + \mathcal{O}\left(\frac{1}{\lambda^3}\right), \quad (32)$$

$$\Pi((\lambda p)^2, q^2; (q + \lambda p)^2) = \frac{1}{\lambda^2} \frac{1}{p^2} \Pi_{VT}(q^2) + \mathcal{O}\left(\frac{1}{\lambda^3}\right). \quad (33)$$

In the following we will use only two-momentum OPE. The reason is that for general correlator, not all the high energy constraints can be simultaneously satisfied using only finite number of resonances in the effective Lagrangian. This statement has been proved in [17] for the case of the $\langle PPS \rangle$ three-point function. For the $\langle VVP \rangle$ this problem has been partially studied in [12] (see also [37]).

By means of an explicit calculation based on $\mathcal{L}_{\text{R}\chi\text{T}}^{(6, \text{odd})}$ (the relevant Feynman graphs are depicted in Fig.1) we get

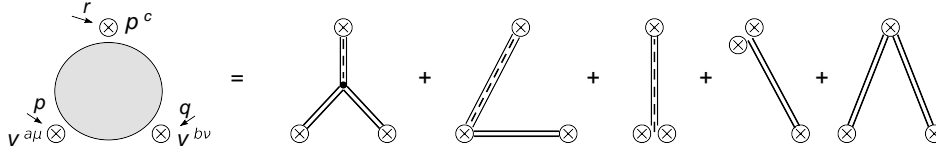


Figure 1: Feynman graphs contributed to VVP Green function. Double lines stand for resonances and dash lines for GB (double lines together with dash lines is the sum of both possible contributions). The crossing is implicitly assumed.

$$\frac{1}{B_0} \Pi^{\text{R}\chi\text{T}}(p^2, q^2, r^2) =$$

$$\begin{aligned}
&= -\frac{N_C}{16\pi^2 r^2} + \frac{4F_V^2 \kappa_3^{VV} p^2}{r^2(p^2 - M_V^2)(q^2 - M_V^2)} - \frac{16\sqrt{2}d_m F_V \kappa_3^{PV}}{(p^2 - M_V^2)(r^2 - M_P^2)} - \frac{32d_m \kappa_5^P}{r^2 - M_P^2} \\
&- \frac{8d_m F_V^2 \kappa^{VVP}}{(p^2 - M_V^2)(q^2 - M_V^2)(r^2 - M_P^2)} + \frac{2F_V^2}{(p^2 - M_V^2)(q^2 - M_V^2)} [8\kappa_2^{VV} - \kappa_3^{VV}] \\
&- \frac{2\sqrt{2}F_V}{r^2(p^2 - M_V^2)} [p^2(\kappa_{16}^V + 2\kappa_{12}^V) - q^2(\kappa_{16}^V - 2\kappa_{17}^V + 2\kappa_{12}^V) - r^2(8\kappa_{14}^V + \kappa_{16}^V + 2\kappa_{12}^V)] \\
&+ (p \leftrightarrow q).
\end{aligned} \tag{34}$$

From OPE (31) we get then the following constraints for the couplings

$$\begin{aligned}
\kappa_{14}^V &= \frac{N_C}{256\sqrt{2}\pi^2 F_V}, \quad \kappa_{16}^V + 2\kappa_{12}^V = -\frac{N_C}{32\sqrt{2}\pi^2 F_V}, \quad \kappa_{17}^V = -\frac{N_C}{64\sqrt{2}\pi^2 F_V}, \quad \kappa_5^P = 0, \\
\kappa_2^{VV} &= \frac{F^2 + 16\sqrt{2}d_m F_V \kappa_3^{PV}}{32F_V^2} - \frac{N_C M_V^2}{512\pi^2 F_V^2}, \quad 8\kappa_2^{VV} - \kappa_3^{VV} = \frac{F^2}{8F_V^2}.
\end{aligned} \tag{35}$$

By employing these constraints one gets²

$$\begin{aligned}
\frac{1}{B_0} \Pi^{\text{R}\chi\text{T}}(p^2, q^2; r^2) &= \frac{F^2}{2} \frac{(p^2 + q^2 + r^2) - \frac{N_C M_V^4}{4\pi^2 F^2}}{(p^2 - M_V^2)(q^2 - M_V^2)r^2} - \frac{16d_m F_V^2 \kappa^{VVP}}{(r^2 - M_P^2)(p^2 - M_V^2)(q^2 - M_V^2)} \\
&- \frac{16\sqrt{2}d_m F_V \kappa_3^{PV} [(p^2 + q^2)M_P^2 - 2r^2 M_V^2]}{r^2(r^2 - M_P^2)(q^2 - M_V^2)(p^2 - M_V^2)}.
\end{aligned} \tag{36}$$

This should be equivalent with the LMD+P ansatz introduced in [11] so that two independent constants b and c introduced there are directly connected with phenomenological couplings κ_3^{PV} and κ^{VVP} . Considering just vector resonance interactions, $\kappa^{VVP} = \kappa_3^{PV} = 0$ (or equivalently taking the limit $M_P \rightarrow \infty$ in (34)), we can reconstruct the LMD ansatz [12]

$$\frac{1}{B_0} \Pi^{\text{LMD}}(p^2, q^2; r^2) = \frac{F^2}{2} \cdot \frac{(p^2 + q^2 + r^2) - \frac{N_C M_V^4}{4\pi^2 F^2}}{(p^2 - M_V^2)(q^2 - M_V^2)r^2}. \tag{37}$$

The result in ChPT up to $\mathcal{O}(p^6)$ at the leading order in $1/N_C$ expansion includes two LECs from the $\mathcal{O}(p^6)$ anomalous sector

$$\frac{1}{B_0} \Pi^{\text{ChPT}}(p^2, q^2; r^2) = -\frac{N_C}{8\pi^2 r^2} + 32C_7^W - \frac{8C_{22}^W(p^2 + q^2)}{r^2}. \tag{38}$$

Comparing this with a low energy expansion of the $R\chi T$ result (34) we give the following lowest-resonance contribution to C_7^W and C_{22}^W (cf. also Section 5)

$$\begin{aligned}
C_7^W &= \frac{F_V^2(8\kappa_2^{VV} - \kappa_3^{VV})}{8M_V^4} + \frac{d_m F_V^2 \kappa^{VVP}}{2M_P^2 M_V^4} - \frac{\sqrt{2}d_m F_V \kappa_3^{PV}}{M_P^2 M_V^2} + \frac{2d_m \kappa_5^P}{M_P^2} - \frac{F_V(2\kappa_{12}^V + 8\kappa_{14}^V + \kappa_{16}^V)}{4\sqrt{2}M_V^2}, \\
C_{22}^W &= -\frac{F_V \kappa_{17}^V}{\sqrt{2}M_V^2} - \frac{F_V^2 \kappa_3^{VV}}{2M_V^4}.
\end{aligned} \tag{39}$$

²Note that these constraints imply automatically also the fulfilment of (33). However, the requirement (32) cannot be satisfied until $\kappa_3^{PV} = 0$, which is in contradiction with another high-energy constraint for related pion transition form factor; see next subsection (cf. also [12]).

Using the OPE constraints (35) we obtain

$$\begin{aligned} C_7^W &= \frac{F^2}{64M_V^4} + \frac{d_m F_V (-2\sqrt{2}\kappa_3^{PV} M_V^2 + F_V \kappa^{VVP})}{2M_P^2 M_V^4}, \\ C_{22}^W &= -\frac{F^2}{16M_V^4} + \frac{N_C}{64\pi^2 M_V^2} - \frac{2\sqrt{2}d_m F_V \kappa_3^{PV}}{M_V^4}. \end{aligned} \quad (40)$$

4.1.1 Formfactors

Let us define fully off-shell formfactors for $\mathcal{P}^* \gamma^* \gamma^*$ vertex, where \mathcal{P} can represents either pion (or any other Goldstone boson) or pseudoscalar resonance via

$$\mathcal{F}_{\mathcal{P}\gamma\gamma}(p^2, q^2; r^2) = \frac{1}{\mathcal{Z}_{\mathcal{P}}}(r^2 - m_{\mathcal{P}}^2)\Pi(p^2, q^2; r^2), \quad (41)$$

where \mathcal{Z} factor interpolates between pseudoscalar source and \mathcal{P} . Let us discuss in detail the $\pi^0 \gamma \gamma$ formfactor. We have $\mathcal{Z}_{\pi^0} = 3/2BF$ and using the OPE constraints (35) we can define (note we are working in the chiral limit)

$$\mathcal{F}_{\pi^0 \gamma \gamma}^{\text{R}\chi\text{T}}(p^2, q^2; r^2) = \frac{2}{3} \frac{1}{BF} r^2 \Pi^{\text{R}\chi\text{T}}(p^2, q^2; r^2), \quad (42)$$

where $\Pi^{\text{R}\chi\text{T}}(p^2, q^2; r^2)$ was introduced in (36). For on-shell pion the κ^{VVP} drops out (note that this is not connected with the chiral limit simplification) and we get a simple result

$$\mathcal{F}_{\pi^0 \gamma \gamma}^{\text{R}\chi\text{T}}(p^2, q^2; 0) = \frac{F}{3} \frac{(p^2 + q^2)(1 + 32\sqrt{2}\frac{d_m F_V}{F^2}\kappa_3^{PV}) - \frac{N_C}{4\pi^2}\frac{M_V^4}{F^2}}{(p^2 - M_V^2)(q^2 - M_V^2)}. \quad (43)$$

Dropping κ_3^{PV} we can again reconstruct the LMD ansatz

$$\mathcal{F}_{\pi^0 \gamma \gamma}^{\text{R}\chi\text{T}}(p^2, q^2; 0) \Big|_{\kappa_3^{PV}=0} = \mathcal{F}_{\pi^0 \gamma \gamma}^{\text{LMD}}(p^2, q^2; 0) = \frac{F_\pi}{3} \frac{p^2 + q^2 - \frac{N_C}{4\pi^2}\frac{M_V^4}{F_\pi^2}}{(p^2 - M_V^2)(q^2 - M_V^2)}. \quad (44)$$

Using Brodsky-Lepage (B-L) behaviour for large momentum [38] [39]

$$\text{B-L cond.} \quad \lim_{Q^2 \rightarrow \infty} \mathcal{F}_{\pi^0 \gamma \gamma}(0, -Q^2; m_\pi^2) \sim -\frac{1}{Q^2}, \quad (45)$$

one can arrive to the following constraint

$$\text{B-L cond.} \quad \kappa_3^{PV} = -\frac{F^2}{32\sqrt{2}d_m F_V}. \quad (46)$$

Before discussing the possible violation of the Brodsky-Lepage condition let us study the influence of the constraint (46) on the original VVP Green function. The $\Pi^{\text{R}\chi\text{T}}$ correlator in (36) will now depend only on one constant κ^{VVP} and we get

$$\frac{1}{B_0} \Pi^{\text{R}\chi\text{T}}(p^2, q^2; r^2) = \frac{F^2}{2} \frac{(p^2 + q^2 + r^2) - \frac{N_C M_V^4}{4\pi^2 F^2}}{(p^2 - M_V^2)(q^2 - M_V^2)r^2} + \frac{F^2}{2} \frac{[(p^2 + q^2)M_P^2 - 2r^2 M_V^2]}{r^2(r^2 - M_P^2)(q^2 - M_V^2)(p^2 - M_V^2)}$$

$$- \frac{16d_m F_V^2 \kappa^{VVP}}{(r^2 - M_P^2)(p^2 - M_V^2)(q^2 - M_V^2)}. \quad (47)$$

The violation of the OPE (32) is here manifest. The remaining constant κ^{VVP} will drop out for an on-shell pion in the formfactor and one gets:

$$\text{B-L cond.:} \quad \mathcal{F}_{\pi^0 \gamma \gamma}^{\text{R}\chi\text{T}}(p^2, q^2; 0) = -\frac{N_C}{12\pi^2 F} \frac{M_V^4}{(p^2 - M_V^2)(q^2 - M_V^2)}. \quad (48)$$

For completeness let us also provide the ChPT result. From (38) using (40) we have

$$\mathcal{F}_{\pi^0 \gamma \gamma}^{\text{ChPT}}(p^2, q^2; 0) = -\frac{N_C}{12\pi^2 F} \left[1 + \frac{p^2 + q^2}{M_V^2} \left(1 - \frac{4\pi^2 F^2}{N_C M_V^2} (1 + 32\sqrt{2} \frac{d_m F_V}{F^2} \kappa_3^{PV}) \right) \right] \quad (49)$$

and

$$\text{B-L cond.:} \quad \mathcal{F}_{\pi^0 \gamma \gamma}^{\text{ChPT}}(p^2, q^2; 0) = -\frac{N_C}{12\pi^2 F} \left(1 + \frac{p^2 + q^2}{M_V^2} \right). \quad (50)$$

For a reader's convenience let us also summarize previous results based on the VMD and LMD+V ansätze³ [12]:

$$\mathcal{F}_{\pi^0 \gamma \gamma}^{\text{VMD}}(p^2, q^2; 0) = -\frac{N_C}{12\pi^2 F_\pi} \frac{M_V^2}{(p^2 - M_V^2)} \frac{M_V^2}{(q^2 - M_V^2)}, \quad (51)$$

$$\mathcal{F}_{\pi^0 \gamma \gamma}^{\text{LMD+V}}(p^2, q^2; 0) = \frac{F_\pi}{3} \frac{p^2 q^2 (p^2 + q^2) + h_1 (p^2 + q^2)^2 + h_2 p^2 q^2 + h_5 (p^2 + q^2) + h_7}{(p^2 - M_{V_1}^2)(p^2 - M_{V_2}^2)(q^2 - M_{V_1}^2)(q^2 - M_{V_2}^2)}, \quad (52)$$

with (valid in the chiral limit)

$$h_7 = -\frac{N_C}{4\pi^2} \frac{M_{V_1}^4 M_{V_2}^4}{F_\pi^2}.$$

We have therefore the following relation (compare with (44))

$$\mathcal{F}_{\pi^0 \gamma \gamma}^{\text{R}\chi\text{T}}(p^2, q^2; 0) \Big|_{\text{B-L}} = \mathcal{F}_{\pi^0 \gamma \gamma}^{\text{VMD}}(p^2, q^2; 0). \quad (53)$$

Now let us turn back to Brodsky-Lepage condition. We have seen it has important consequences on the actual form of the $\pi^0 - \gamma - \gamma$ formfactor within R χ T. However, recent BABAR measurement [40] showed phenomenological disagreement with this condition. There are also theoretical arguments [17] which showed that high-energy constraints cannot be all satisfied for a given formfactor within the ansatz with only finite number of resonance poles. (For a recent study on Brodsky-Lepage revision see [41]; see also [42] and references therein.) We will thus relax the Brodsky-Lepage condition by allowing a small deviation from (46) parameterized with δ_{BL}

$$\kappa_3^{PV} = -\frac{F^2}{32\sqrt{2}d_m F_V} (1 + \delta_{\text{BL}}). \quad (54)$$

Its actual value can be set by fitting the BABAR and CLEO data. In this fit and also in the following phenomenological applications we set

$$M_V = m_\rho \approx 0.775 \text{ GeV}, \quad M_P = m_{\pi'} \approx 1.3 \text{ GeV}, \quad F = F_\pi \approx 92.22 \text{ MeV} \quad (55)$$

³The LMD+V ansatz adds one extra vector multiplet in comparison with the LMD one. This corresponds to MHA for which all the OPE and B-L constraints can be satisfied simultaneously.

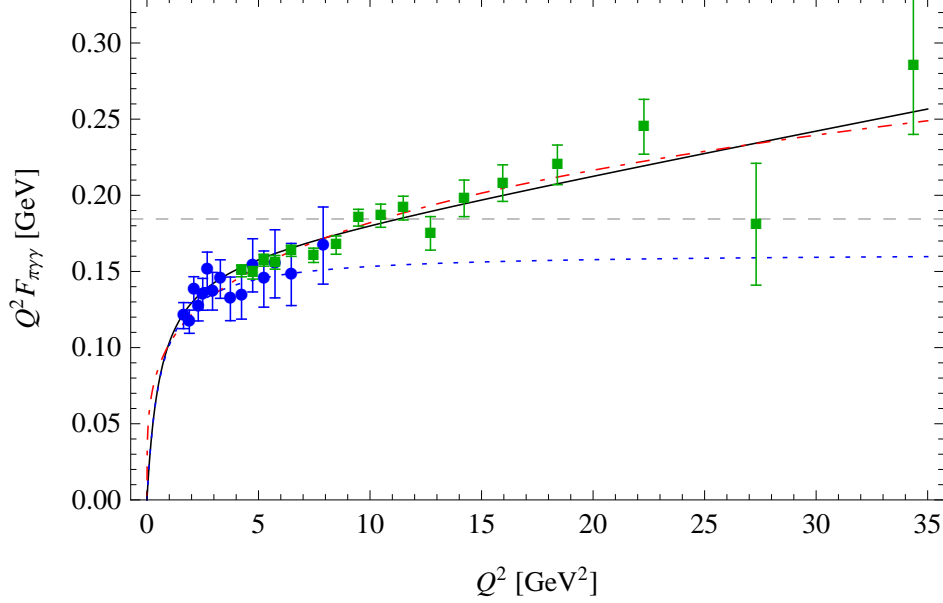


Figure 2: CLEO (blue points) and BABAR (green squares) data with fitted function $\mathcal{F}_{\pi^0\gamma\gamma}^{\text{R}\chi\text{T}}(0, -Q^2; 0)$ defined in (43) using the modified Brodsky-Lepage condition in (54). The full line is for $\delta_{\text{BL}} = -0.055$ and (blue) dotted line stands for standard B-L (i.e. $\delta_{\text{BL}} = 0$). Dot-dashed (red) line shows fitted function $A(Q^2/(10 \text{ GeV}^2))^\beta$ with $A = 0.182 \pm 0.002 \text{ GeV}$ and $\beta = 0.25 \pm 0.02$ as obtained by BABAR collaboration [40]. The asymptotic $2F$ is represented by the horizontal dash line.

and also (for details see [43])

$$F_V = F_\rho = 146.3 \pm 1.2 \text{ MeV}, \quad d_m \approx 26 \text{ MeV}. \quad (56)$$

The new BABAR data indicates important negative shift in δ_{BL} with the result

$$\delta_{\text{BL}} = -0.055 \pm 0.025. \quad (57)$$

Our fit together with CLEO and BABAR data is depicted in Fig. 2.

4.1.2 Decay $\rho \rightarrow \pi\gamma$

In this subsection we illustrate a particular phenomenological application of the above results, namely a prediction for $\rho \rightarrow \pi\gamma$ decay. For this process we can use a connection with the off-shell $\pi\gamma\gamma$ formfactor introduced in the previous subsection. First, let us define the amplitude \mathcal{A} for the process $\rho^+(p) \rightarrow \pi^+(p)\gamma(k)$ (we will use only the charged decay process to avoid the discussion on $\omega - \rho$ mixing for the neutral one):

$$\Gamma_{\rho \rightarrow \pi\gamma} = \frac{1}{8\pi} \frac{1}{3} \sum_{\text{pol.}} |\mathcal{A}_{\rho \rightarrow \pi\gamma} \varepsilon^{\mu\nu\alpha\beta} p_\alpha q_\beta \epsilon_\mu(p) \epsilon_\nu^*(k)|^2 \frac{m_\rho^2 - m_\pi^2}{2m_\rho^3}, \quad (58)$$

from which we have already factorized out the Levi-Civita and momentum dependence. Similarly one can define the amplitude for $\pi^0(p) \rightarrow \gamma(k)\gamma(l)$

$$\Gamma_{\pi^0 \rightarrow \gamma\gamma} = \frac{1}{32\pi} \sum_{\text{pol}} |\mathcal{A}_{\pi^0 \rightarrow \gamma\gamma} \varepsilon^{\mu\nu\alpha\beta} k_\alpha l_\beta \epsilon_\mu^*(k) \epsilon_\nu^*(l)|^2 \frac{1}{m_{\pi^0}}. \quad (59)$$

The connection with $\pi\gamma\gamma$ formfactor is obtained via

$$\mathcal{A}_{\rho \rightarrow \pi\gamma} = \frac{e}{2 F_V M_V} \lim_{q^2 \rightarrow M_V^2} (q^2 - M_V^2) \mathcal{F}_{\pi\gamma\gamma}(0, q^2; 0) \quad (60)$$

and quite simply

$$\mathcal{A}_{\pi \rightarrow \gamma\gamma} = e^2 \mathcal{F}_{\pi\gamma\gamma}(0, 0; 0). \quad (61)$$

Putting these two definitions together we can extract the ratio and corresponding parameter x [11]:

$$\frac{2eF_V}{M_V} \left| \frac{\mathcal{A}_{\rho \rightarrow \pi\gamma}}{\mathcal{A}_{\pi \rightarrow \gamma\gamma}} \right| \equiv 1 + x. \quad (62)$$

Using the experimental value $\Gamma_{\rho \rightarrow \pi\gamma} = 68 \pm 7 \text{ keV}$ this parameter was obtained to be equal to $x = 0.022 \pm 0.051$ in [11] based on the 1992 edition of the particle data book (same number was also used later e.g. in [12]). Updating this prediction with a new experimental input we can get flip in the sign

$$\text{exp:} \quad x = -0.003 \pm 0.054. \quad (63)$$

The change is mainly due to a new value of F_V (study e.g. in [43]) and a new precise measurement of π^0 lifetime by PrimEx group [44] (see also [30]).

Within our formalism, the parameter x defined above is proportional to the deviation from the simple VMD ansatz (51), or in other words from the exact Brodsky-Lepage condition (cf. (53)). Using (43) and (54) we get in terms of δ_{BL}

$$x = \frac{4\pi^2 F^2}{M_V^2 N_C} \delta_{BL}. \quad (64)$$

The results of the previous subsection allows us to make rather precise determination of this value

$$\text{R}\chi\text{T:} \quad x = -0.010 \pm 0.005, \quad (65)$$

which using (62) and experimental input for $\Gamma_{\pi^0 \rightarrow \gamma\gamma}$ leads to the following prediction:

$$\text{R}\chi\text{T:} \quad \Gamma_{\rho \rightarrow \pi\gamma} = 67.0 \pm 2.3 \text{ keV}. \quad (66)$$

4.1.3 Decays of $\pi(1300)$

In the previous section we have obtained a prediction for the $\rho \rightarrow \pi\gamma$ decay width. However, it was based on the ratio of two decay widths (cf. (62)) and experimental input of one of them. We could predict also the absolute value for $\rho \rightarrow \pi\gamma$ directly from (58) and (60) without the necessity to use the experimental value of $\pi^0 \rightarrow \gamma\gamma$ (in fact we will discuss a little the latter process in the very next subsection) but one should remember that we have been making several simplifications, namely: we are working in large N_C , using only lowest-lying resonances and we are in the chiral limit. All together within this approximation we cannot

expect the accuracy of the result being better than 30%-40%. On the other hand one can expect that some of the systematic uncertainties will cancel out in the ratios similar to one studied in the previous part.

The same strategy can be repeated for $\pi(1300)$ decays. In fact we can work in exact correspondence; the two decays would be now: $\pi(1300) \rightarrow \rho\gamma$ and $\pi(1300) \rightarrow \gamma\gamma$. The only problem now is that none of these two processes have been seen so far. The most recent limit on $\pi(1300) \rightarrow \gamma\gamma$ by Belle collaboration [45]

$$\Gamma_{\pi' \rightarrow \gamma\gamma} < 72 \text{ eV} \quad (67)$$

sets at least rough limit in our studies. Using the definition (41) the main object here is

$$\mathcal{F}_{P\gamma\gamma}^{\text{R}\chi\text{T}}(p^2, q^2; m_P^2) = \frac{8\sqrt{2}}{3} F_V \frac{\sqrt{2}\kappa_3^{PV}(2M_V^2 - p^2 - q^2) - F_V\kappa^{VVP}}{(p^2 - M_V^2)(q^2 - M_V^2)}. \quad (68)$$

The amplitude for $\pi(1300) \rightarrow \gamma\gamma$ is given by

$$\mathcal{A}_{\pi' \rightarrow \gamma\gamma} = e^2 \mathcal{F}_{P\gamma\gamma}(0, 0; m_{\pi'}^2) \quad (69)$$

and similarly for $\pi(1300) \rightarrow \rho\gamma$ (see also (60)). Then

$$\mathcal{A}_{\pi' \rightarrow \gamma\gamma}^{\text{R}\chi\text{T}} = e^2 \frac{8\sqrt{2}}{3} F_V \frac{2\sqrt{2}\kappa_3^{PV} M_V^2 - F_V\kappa^{VVP}}{M_V^4}, \quad (70)$$

$$\mathcal{A}_{\pi' \rightarrow \rho\gamma}^{\text{R}\chi\text{T}} = -e \frac{4\sqrt{2}}{3M_V} \frac{\sqrt{2}\kappa_3^{PV} M_V^2 - F_V\kappa^{VVP}}{M_V^2}. \quad (71)$$

Both these amplitudes depend on one so far undetermined constant κ^{VVP} . Provided we have experimental values of both branching ratios we could verify the consistency of our model. In the present situation we can visualize how one decay mode depends on the second one, and this was done in Fig. 3. One can see that we have two solutions for κ^{VVP} as we have quadratic equation for decay width as a function of κ^{VVP} and none of these two solutions can be ruled out. Note that the full width for $\pi(1300)$ is assumed to be between 200 and 600 MeV (see [46]), so both processes are extremely suppressed for any of these two solutions.

The experimental bound on $\Gamma_{\pi' \rightarrow \gamma\gamma}$ can be used to get estimate of κ^{VVP} . In order to fulfill the limit (67) we expect the numerator in (68) to be suppressed. This expected suppression leads in analogy with (54) to the following ansatz

$$\kappa_{VVP} = -\frac{F^2 M_V^2}{16d_m F_V^2} (1 + \delta_A), \quad (72)$$

with parameter δ_A which should be reasonably small. In terms of this parameter we get the decay width in a compact form

$$\Gamma_{\pi' \rightarrow \gamma\gamma} = \left(\frac{\alpha F^2}{6\sqrt{2}d_m M_V^2} \right)^2 \pi m_{\pi'}^3 (\delta_{\text{BL}} - \delta_A)^2 \approx (1514.0 \text{ eV}) \times (\delta_{\text{BL}} - \delta_A)^2 \quad (73)$$

and thus the extreme phenomenological suppression of $\pi(1300) \rightarrow \gamma\gamma$ can be understood within our formalism to be due to the small factor $(\delta_{\text{BL}} - \delta_A)^2$. The experimental limit together with (57) set the allowed range for the parameter δ_A

$$-0.27 \lesssim \delta_A \lesssim 0.16, \quad (74)$$

which is good enough to set the value of κ_{VVP} to

$$\kappa_{VVP} \approx (-0.57 \pm 0.13) \text{ GeV}. \quad (75)$$

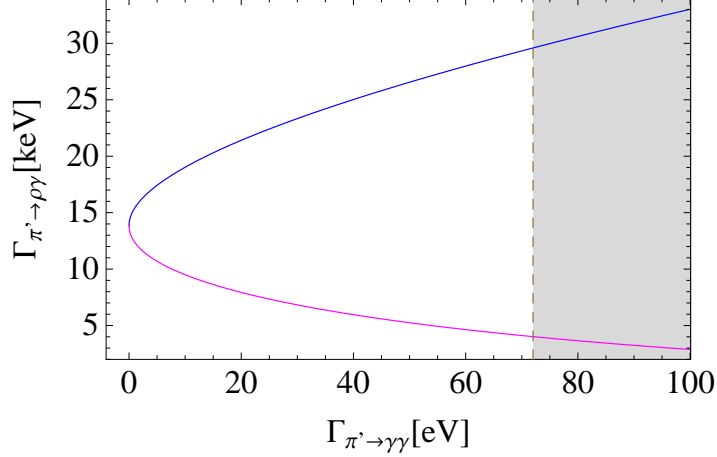


Figure 3: The connection of decay width for $\pi(1300) \rightarrow \gamma\gamma$ and $\pi(1300) \rightarrow \rho\gamma$ (note that we have two possible solutions). The dashed line represents the Belle's limit (67) on $\Gamma_{\pi' \rightarrow \rho\pi}$ (grey area is thus excluded by this experiment).

4.1.4 Decay $\pi^0 \rightarrow \gamma\gamma$ and $\eta \rightarrow \gamma\gamma$

As we have stated, the absolute decay widths are accessible via our approach only with the limited precision. For instance for the $\pi^0 \rightarrow \gamma\gamma$ amplitude we have obtained only very simple prediction (61). It turns out, however, that it agrees very well with the experimental determination. On the other hand, similar determination for $\eta \rightarrow \gamma\gamma$ would be a phenomenological disaster.

In order to go beyond the leading order we can use the chiral corrections calculated using ChPT. The most recent study of $\pi^0 \rightarrow \gamma\gamma$ amplitude went up to NNLO [30]. The motivation for going beyond NLO lays in the fact that there are no chiral logarithms at NLO [47, 48]. At NNLO these logarithms though non-zero are relatively small so the C_i^W play very important role. We can therefore use existing calculations within ChPT with our estimate (40) of C_i^W . Here our approximation, namely the chiral limit, does not make any difference as by construction LECs (C_i^W in our case) do not depend on light quark masses. With previous phenomenological determination of the couplings κ_3^{PV} and κ_{VVP} we obtain

$$C_7^W = \frac{F^2}{64M_V^4} \left(1 + 2 \frac{M_V^2}{M_P^2} (\delta_{BL} - \delta_A) \right) \approx (0.35 \pm 0.07) \times 10^{-3} \text{ GeV}^{-2}. \quad (76)$$

The second and last unknown LEC at NLO for $\pi^0 \rightarrow \gamma\gamma$ and $\eta \rightarrow \gamma\gamma$ is C_8^W . Anticipating the result of the next section and using the OPE constraints (35) we get

$$C_8^W = \frac{N_C}{768M_0^2\pi^2} - \frac{N_C}{512\pi^2M_V^2} - \frac{F_V\kappa_{13}^V}{\sqrt{2}M_V^2} + \frac{F_V^2\kappa_1^{VV}}{2M_V^4} + \frac{d_{m0}F^2}{96d_mM_P^2M_V^2} + \frac{d_{m0}F_V^2\kappa_{VVP}}{6M_P^2M_V^4}, \quad (77)$$

where we have also dropped the term proportional to δ_{BL} because it is not numerically relevant. Unfortunately at this moment similarly as for already mentioned d_{m0} we cannot make an estimate for κ_{13}^V and κ_1^{VV} (all these couplings are dominated by the η' exchange, cf. Appendix A). We may however again connect two-gamma decay widths of π^0 and η . We

may for example set the unknown C_8^W from the experimental value of $\Gamma(\eta \rightarrow \gamma\gamma)$. This was done for NLO $\eta \rightarrow \gamma\gamma$ expression in [30]. There is ongoing project which should enlarge this calculation to the NNLO within ChPT (for preliminary results in the chiral limit calculation see [49]). We thus rather postpone as a future project the final determination of $\pi^0 \rightarrow \gamma\gamma$ based on the experimental value $\Gamma_{\eta \rightarrow \gamma\gamma}$. Let us only mention, that if we assume that the NNLO corrections for η are indeed small as for $\pi^0 \rightarrow \gamma\gamma$, the value in (76) has roughly the influence at 0.5% level for $\Gamma_{\pi^0 \rightarrow \gamma\gamma}$ (with the opposite sign). A new study of isospin breaking effects in [50] indicates another shift of the similar size (however now with a positive sign) and thus at this moment we do not expect quantitative change of the prediction made in [30].

4.1.5 $g - 2$

Probably the main motivation for studying the VVP correlator is hidden in the determination of the muon $g - 2$ factor. It is beyond the scope of this paper to discuss this problem in great detail (for details see e.g. [51]). Let us only mention that the main source of theoretical error for its standard model prediction comes from hadronic contributions, more precisely from the hadronic light-by-light scattering which cannot be related to any available data. The hadronic four-point correlator $VVVV$ is further simplified into three classes of contributions: a) π^\pm and K^\pm loops b) π^0, η, η' exchanges and finally c) the rest, which is modelled usually via constituent quark loops. It is clear that this separation is not without ambiguity and different approaches can differently calculate contribution especially between a) and c) or b) and c). Our contribution based on the VVP correlator study belongs to the group b). To avoid inconsistency we will work in close relation with similar work done for LMD or VMD ansätze [52]. Using the fully off-shell (i.e. including also the π^0 off-shellness) $\pi^0 - \gamma - \gamma$ formfactor (42) we arrive to

$$a_\mu^{\text{LbyL}; \pi^0} = (65.8 \pm 1.2) \times 10^{-11}. \quad (78)$$

In the error only the uncertainties of our model were included. The systematic is mainly influenced by the above mentioned ambiguity of how one defines and splits the pion-pole and regular part from the $\langle VVVV \rangle$. We have put the cutoff energy at 10 GeV. For a better comparison let us present in Table 8 predictions for the studied contribution to the muon $g - 2$ for the different models summarized in (52). We have recalculated there the light-by-

model	$a_\mu^{\text{LbyL}; \pi^0} \times 10^{11}$
VMD	57.2
LMD	73.7
LMD+V “on-shell”	58.2
LMD+V “off-shell”	72 ± 12
this work	65.8 ± 1.2

Table 8: Contribution of π^0 exchange to the muon $g - 2$ factor for different models.

light contributions based on VMD and LMD ansätze. We have also reevaluated the case of LMD+V ansatz or more precisely its on-shell simplification as defined in (52). Three unknown constants are set similarly as done in [12], i.e. $h_{1,2} = 0$ and h_5 is based on the $\rho \rightarrow \pi\gamma$ phenomenology $h_5 = 6.99$ (obtained for the updated value in (63)). The full LMD+V

“off-shell” ansatz has 7 parameters (for details see [37]). One relation can be obtained from the chiral anomaly and others can be: Brodsky-Lepage behaviour, higher-twist corrections in the OPE and one-large momentum OPE, together with data (CLEO for this turn) we are still left with two undetermined parameters. Their variations in reasonable range set the final error for the corresponding LMD+V value in Tab. 8. Let us note that also the possibility of B-L violation together with new fit of two parameters (h_1 and h_5) was studied in [37] with no influence on the central value of $g - 2$ contribution. Too many parameters is not the only problem connected with the LMD+V ansatz. Status of $\rho(1450)$ as a first radial excitation of $\rho(770)$ is doubted by the possible existence of lighter $\rho(1250)$ [53]. Its presence is also supported by the study within AdS/QCD approaches [54]. On top of that the inclusion of the complete set of all excitations in all channels (i.e. inclusion of π'') can change again the studied ansatz similarly as we have encountered for the first excitation (see (44) and (53)).

Let us also note quite astonishing coincidence of our result with the most recent study based on AdS/QCD conjecture [55] $a_\mu^{\pi^0} = 65.4(2.5) \times 10^{-11}$.

4.2 VAS Green function

The $\langle VVP \rangle$ Green’s function studied in the previous section represents without any doubts the most important example of the odd intrinsic sector of QCD. However, it is not the only quantity one can analyze using our complete lowest-lying resonance model. As the second example we have chosen $\langle VAS \rangle$, which has not yet been studied (to our knowledge) in the literature. It also enables to demonstrate the use of the “second half” our the odd intrinsic resonance Lagrangian, i.e. those with 1^{++} and 0^{++} states.

Defining (beware of the same symbol as for $\langle VVP \rangle$)

$$\Pi_{\mu\nu}^{abc}(p, q) = \int d^4x d^4y e^{ip \cdot x + iq \cdot y} \langle 0 | T [V_\mu^a(x) A_\nu^b(y) S^c(0) | 0] \rangle, \quad (79)$$

with (cf. also (29))

$$A_\mu^a(x) = \bar{q}(x) \gamma_\mu \gamma_5 \frac{\lambda^a}{2} q(x), \quad S_\mu^a(x) = \bar{q}(x) \frac{\lambda^a}{2} q(x).$$

Similarly as for VVP one can write

$$\Pi(p, q)_{\mu\nu}^{abc} = f^{abc} \epsilon_{\mu\nu\alpha\beta} p^\alpha q^\beta \Pi(p^2, q^2; r^2), \quad (80)$$

where $r = -(p + q)$. In resonance region, we have

$$\begin{aligned} \frac{1}{B_0} \Pi(p^2, q^2; r^2) &= \frac{8\sqrt{2}F_V(\kappa_4^V - 2\kappa_{15}^V)}{p^2 - M_V^2} + \frac{16\sqrt{2}F_A\kappa_{14}^A}{q^2 - M_A^2} + \frac{32c_m\kappa_2^S}{r^2 - M_S^2} + \frac{16\sqrt{2}F_Ac_m\kappa_1^{SA}}{(q^2 - M_A^2)(r^2 - M_S^2)} \\ &- \frac{8\sqrt{2}F_Vc_m(2\kappa_1^{SV} + \kappa_2^{SV})}{(p^2 - M_V^2)(r^2 - M_S^2)} - \frac{16F_AF_V\kappa_6^{VA}}{(q^2 - M_A^2)(p^2 - M_V^2)} + \frac{16F_AF_Vc_m\kappa^{VAS}}{(q^2 - M_A^2)(p^2 - M_V^2)(r^2 - M_S^2)}. \end{aligned} \quad (81)$$

At high energies one can obtain the following OPE relation

$$\Pi((\lambda p)^2, (\lambda q)^2; (\lambda r)^2) = \frac{B_0 F^2}{2\lambda^4} \frac{p^2 - q^2 - r^2}{p^2 q^2 r^2} + \mathcal{O}\left(\frac{1}{\lambda^6}\right) \quad (82)$$

and again we will not consider here one-momentum OPE limits. The high-energy constraint leads to

$$\begin{aligned}\kappa_2^S = \kappa_{14}^A = 0, \quad \kappa_4^V = 2\kappa_{15}^V, \quad \kappa_6^{VA} = \frac{F^2}{32F_A F_V}, \\ F_V(2\kappa_1^{SV} + \kappa_2^{SV}) = 2F_A \kappa_1^{SA} = \frac{F^2}{16\sqrt{2}c_m}.\end{aligned}\quad (83)$$

If we use these relations, we have finally only one free parameter: κ^{VAS} ; the result is

$$\frac{1}{B_0}\Pi^{\text{R}\chi\text{T}}(p^2, q^2; r^2) = \frac{F^2(p^2 - q^2 - r^2 - M_V^2 + M_A^2 + M_S^2) + 32F_A F_V c_m \kappa^{VAS}}{2(q^2 - M_A^2)(p^2 - M_V^2)(r^2 - M_S^2)}.\quad (84)$$

From the theoretical point of view we are thus in a better position than we were for $\langle VVP \rangle$. After imposing OPE we are left with one free parameter whereas in the case of $\langle VVP \rangle$ we had two (cf. (36)). We can thus simply connect all processes schematically represented as

$$(V : \rho, \omega, K^*, \gamma, \dots) \sim (A : a_1, f_1, K_1, GB, W \dots) \sim (S : \sigma, \kappa, a_0, f_0, H \dots) \quad (85)$$

via a single parameter. The problem is that they are very rare and have not yet been measured, on top of that the status of some of the particle content is controversial by itself (especially if talking about a scalar sector). The parameter κ^{VAS} can be, however, set using other (not that rare) processes it enters. One way is to check in next section to which of 23 parameters it contributes and use directly the system of LECs. This can be done already here within the calculation of VAS. At low energies, up to $\mathcal{O}(p^6)$ one has

$$\frac{1}{B_0}\Pi(p^2, q^2, r^2) = -32C_{11}^W.\quad (86)$$

Comparing with the low energy expansion of the full R χ T result (81) we get

$$\begin{aligned}C_{11}^W &= \frac{F_A \kappa_{14}^A}{\sqrt{2}M_A^2} + \frac{F_V(\kappa_4^V - 2\kappa_{15}^V)}{2\sqrt{2}M_V^2} + \frac{c_m \kappa_2^S}{M_S^2} + \frac{F_A F_V \kappa_6^{VA}}{2M_A^2 M_V^2} + \frac{c_m F_V(2\kappa_1^{SV} + \kappa_2^{SV})}{2\sqrt{2}M_S^2 M_V^2} \\ &\quad - \frac{F_A c_m \kappa_1^{SA}}{\sqrt{2}M_A^2 M_S^2} + \frac{F_A F_V c_m \kappa^{VAS}}{2M_A^2 M_S^2 M_V^2}.\end{aligned}\quad (87)$$

Using the OPE (83) we obtain

$$C_{11}^W = \frac{F^2}{64} \left[\frac{1}{M_S^2 M_V^2} + \frac{1}{M_A^2 M_V^2} - \frac{1}{M_A^2 M_S^2} \right] + \frac{F_A F_V c_m \kappa^{VAS}}{2M_A^2 M_S^2 M_V^2}.$$

The knowledge of C_{11}^W leads directly to the value of κ^{VAS} and thus to the rare processes schematically specified above. The current attempts for a C_{11}^W estimation were summarized in Table 1 of [56] with rather inconsistent values obtained both from the phenomenology ([57], [58]) and by a model-dependent determination [56]. The most precise value seems to be obtained from $K^+ \rightarrow l^+ \nu \gamma$ data [59]: $C_{11}^W = (0.68 \pm 0.21) \times 10^{-3} \text{ GeV}^{-2}$ [57]. Using the values set in (55) and (56), together with

$$M_S = m_{a_0} \approx 984.7 \text{ MeV}, \quad c_m \approx 42 \text{ MeV} \quad (88)$$

and the Weinberg sum rules (to get values of M_A and F_A) we arrive at

$$\kappa^{VAS} = 0.61 \pm 0.40 \text{ GeV}.\quad (89)$$

4.3 Short note on the field redefinition

The previous two examples were calculated using the full resonance Lagrangian $\mathcal{L}_{R\chi T}^{(6, \text{odd})}$. Here we would like to address a question what would happened if one would repeat the same calculation but instead use the reduced resonance Lagrangian $\overline{\mathcal{L}}_{R\chi T}^{(6, \text{odd})}$. This Lagrangian is established in Appendix B and can be obtained from the full Lagrangian (25) by means dropping the operators marked with a star in the Tables 1-7 i.e. by means of omitting 20 parameters: $\kappa_{1,2}^{RR}, \kappa_i^{SA}, \kappa_1^{SV}, \kappa_i^{VA}, \kappa_1^{PA}, \kappa_i^{PV}, \kappa_i^{RRR}$ and using the bar over the rest of κ_i^X (see Section B.4). This can be motivated by its equivalent contribution to the saturation of LECs. This exercise was already performed in [24] for $\langle VAP \rangle$ with an interesting finding, that after imposing the OPE condition the both results are the same. In our case the conclusion is, however, different. Using the reduced resonance Lagrangian we would not be able to simply fulfill the OPE constraints by imposing some conditions on $\overline{\kappa_i^X}$. In the first case, the OPE for $\langle VVP \rangle$ requires an additional relation, namely $M_V = \frac{4\pi F}{\sqrt{N_C}}$. In the second case, $\langle VAS \rangle$, the OPE cannot be satisfied at all.

Thus we have to conclude that the equivalence of both calculation in the even sector for $\langle VAP \rangle$ was just a coincidence and it is not a general feature.

5 Resonance contributions to the LECs of the anomalous sector

We have seen in the previous two applications the explicit examples of the calculation with the resonance fields. A match between this result in a region of small momenta (i.e. $p \ll M_R$) at one side and the ChPT result at other side enables to extract the dependence of LECs on resonances. In this way we have obtained within VVP calculation C_7^W and C_{22}^W (see (39)) and from VAS it was possible to extract C_{11}^W (87). The dependence of all others C_i^W on the parameters of the resonance model can be obtained by systematic integration-out of all resonances. So obtained Lagrangian can be expand over the canonical basis of NLO odd-intrinsic Lagrangian established for example in [8]. In this way we have saturated 21 of 23 constants and only C_3^W and C_{18}^W stayed intact as they are subleading in large N_C . The η' was explicitly considered (see Appendix A and [60]) and it contributes in C_6^W , C_8^W and C_{10}^W . It is always the first term in these LECs and we put it in the boldface font to stress its large N_C dominance over the rest. Generally we have the following expansion in large N_C for all C_i^W , schematically

$$C_i^W = a_i N_C^2 + b_i N_C + O(N_C^0), \quad (90)$$

where $a_i \neq 0$ for $i = 6, 8, 10$ and $b_i = 0$ for $i = 3, 18$.

The field redefinition similarly as done in [24] was performed and details are summarized in Appendix B. All 20 parameters denoted by stars in Tab.1–7 can be dropped in the following and for all others a bar should be added (bar parameters $\overline{\kappa_i^X}$ are defined in the last section of Appendix B). We prefer, however, to use the original parametrization as it represents direct connection with the resonance phenomenology and is thus simpler to use.

The explicit form of the resonance saturation generated by the resonance Lagrangian (25) is:

$$C_1^W = \frac{d_m \kappa_4^P}{M_P^2} + \frac{2\sqrt{2}d_m G_V \kappa_1^{PV}}{M_P^2 M_V^2} - \frac{\sqrt{2}d_m G_V \kappa_2^{PV}}{M_P^2 M_V^2} + \frac{\sqrt{2}G_V \kappa_3^V}{M_V^2} - \frac{2\sqrt{2}G_V \kappa_9^V}{M_V^2} + \frac{\sqrt{2}G_V \kappa_{10}^V}{M_V^2}$$

$$\begin{aligned}
& -\frac{4G_V^2\kappa_2^{\text{VV}}}{M_V^4} - \frac{3G_V^2\kappa_3^{\text{VV}}}{2M_V^4} - \frac{2\kappa^{\text{VVP}}d_mG_V^2}{M_P^2M_V^4}, \\
C_2^W &= \frac{F_A\kappa_{13}^A}{\sqrt{2}M_A^2} + \frac{c_m\kappa_1^S}{M_S^2} + \frac{c_mF_A\kappa_2^{\text{SA}}}{\sqrt{2}M_A^2M_S^2} + \frac{\sqrt{2}c_mG_V\kappa_1^{\text{SV}}}{M_S^2M_V^2} + \frac{c_mF_V\kappa_2^{\text{SV}}}{2\sqrt{2}M_S^2M_V^2} + \frac{F_V\kappa_4^V}{2\sqrt{2}M_V^2} - \frac{\sqrt{2}G_V\kappa_{15}^V}{M_V^2} \\
& + \frac{F_AG_V\kappa_6^{\text{VA}}}{M_A^2M_V^2} + \frac{\kappa^{\text{VAS}}c_mF_AG_V}{M_A^2M_S^2M_V^2}, \\
C_3^W &= 0, \\
C_4^W &= -\frac{F_A\kappa_3^A}{2\sqrt{2}M_A^2} - \frac{d_m\kappa_3^P}{M_P^2} + \frac{\sqrt{2}d_mG_V\kappa_3^{\text{PV}}}{M_P^2M_V^2} + \frac{F_AF_V\kappa_3^{\text{VA}}}{2M_A^2M_V^2} + \frac{\sqrt{2}F_A\kappa_4^A}{M_A^2} - \frac{F_A\kappa_{12}^A}{2\sqrt{2}M_A^2} + \frac{F_A\kappa_{15}^A}{2\sqrt{2}M_A^2} \\
& - \frac{F_A^2\kappa_4^{\text{AA}}}{4M_A^4} - \frac{d_mF_A\kappa_2^{\text{PA}}}{2\sqrt{2}M_A^2M_P^2} + \frac{d_mF_V\kappa_1^{\text{PV}}}{\sqrt{2}M_P^2M_V^2} + \frac{2c_d\kappa_2^S}{M_S^2} - \frac{\sqrt{2}c_dF_A\kappa_1^{\text{SA}}}{M_A^2M_S^2} + \frac{\sqrt{2}c_dF_V\kappa_1^{\text{SV}}}{M_S^2M_V^2} \\
& + \frac{c_dF_V\kappa_2^{\text{SV}}}{\sqrt{2}M_S^2M_V^2} + \frac{F_V\kappa_1^V}{\sqrt{2}M_V^2} - \frac{\sqrt{2}F_V\kappa_5^V}{M_V^2} - \frac{F_V\kappa_6^V}{\sqrt{2}M_V^2} - \frac{F_V\kappa_8^V}{\sqrt{2}M_V^2} - \frac{F_V\kappa_9^V}{\sqrt{2}M_V^2} + \frac{\sqrt{2}G_V\kappa_{14}^V}{M_V^2} \\
& + \frac{F_AF_V\kappa_1^{\text{VA}}}{M_A^2M_V^2} + \frac{F_AF_V\kappa_2^{\text{VA}}}{2M_A^2M_V^2} - \frac{2F_VG_V\kappa_2^{\text{VV}}}{M_V^4} + \frac{\kappa^{\text{VAS}}c_dF_AF_V}{M_A^2M_S^2M_V^2} - \frac{\kappa^{\text{VVP}}d_mF_VG_V}{M_P^2M_V^4}, \\
C_5^W &= \frac{F_A\kappa_{12}^A}{\sqrt{2}M_A^2} - \frac{d_m\kappa_2^P}{M_P^2} + \frac{d_mF_A\kappa_2^{\text{PA}}}{\sqrt{2}M_A^2M_P^2} - \frac{d_mF_V\kappa_2^{\text{PV}}}{\sqrt{2}M_P^2M_V^2} + \frac{F_V\kappa_{10}^V}{\sqrt{2}M_V^2} - \frac{G_V\kappa_{17}^V}{\sqrt{2}M_V^2} - \frac{F_VG_V\kappa_3^{\text{VV}}}{M_V^4}, \\
C_6^W &= -\frac{\mathbf{N_C}}{576M_0^2\pi^2} + \frac{F_A\kappa_3^A}{3\sqrt{2}M_A^2} - \frac{F_AF_V\kappa_3^{\text{VA}}}{3M_A^2M_V^2} - \frac{F_VG_V\kappa_3^{\text{VV}}}{3M_V^4} - \frac{2\sqrt{2}F_A\kappa_4^A}{3M_A^2} + \frac{F_A\kappa_{10}^A}{\sqrt{2}M_A^2} - \frac{F_A\kappa_{15}^A}{3\sqrt{2}M_A^2} \\
& + \frac{F_A^2\kappa_4^{\text{AA}}}{6M_A^4} - \frac{4c_d\kappa_2^S}{3M_S^2} + \frac{2\sqrt{2}c_dF_A\kappa_1^{\text{SA}}}{3M_A^2M_S^2} - \frac{2\sqrt{2}c_dF_V\kappa_1^{\text{SV}}}{3M_S^2M_V^2} - \frac{\sqrt{2}c_dF_V\kappa_2^{\text{SV}}}{3M_S^2M_V^2} - \frac{\sqrt{2}F_V\kappa_1^V}{3M_V^2} \\
& + \frac{2\sqrt{2}F_V\kappa_5^V}{3M_V^2} + \frac{\sqrt{2}F_V\kappa_6^V}{3M_V^2} + \frac{\sqrt{2}F_V\kappa_8^V}{3M_V^2} + \frac{\sqrt{2}G_V\kappa_{13}^V}{M_V^2} - \frac{G_V\kappa_{17}^V}{3\sqrt{2}M_V^2} - \frac{F_V\kappa_{18}^V}{\sqrt{2}M_V^2} \\
& - \frac{2F_AF_V\kappa_1^{\text{VA}}}{3M_A^2M_V^2} - \frac{F_AF_V\kappa_2^{\text{VA}}}{3M_A^2M_V^2} - \frac{2F_VG_V\kappa_1^{\text{VV}}}{M_V^4} - \frac{2\kappa^{\text{VAS}}c_dF_AF_V}{3M_A^2M_S^2M_V^2} - \frac{\sqrt{2}F_Ad_{m0}\kappa_2^{\text{PA}}}{3M_A^2M_P^2} - \frac{2d_{m0}\kappa_3^P}{3M_P^2} \\
& + \frac{d_{m0}\kappa_2^P}{3M_P^2} + \frac{d_{m0}F_V\kappa_2^{\text{PV}}}{3\sqrt{2}M_P^2M_V^2} + \frac{\sqrt{2}d_{m0}F_V\kappa_1^{\text{PV}}}{3M_P^2M_V^2} + \frac{2\sqrt{2}d_{m0}G_V\kappa_3^{\text{PV}}}{3M_P^2M_V^2} - \frac{2d_{m0}F_VG_V\kappa^{\text{VVP}}}{3M_P^2M_V^4}, \\
C_7^W &= \frac{2d_m\kappa_5^P}{M_P^2} - \frac{\sqrt{2}d_mF_V\kappa_3^{\text{PV}}}{M_P^2M_V^2} - \frac{F_V\kappa_{12}^V}{2\sqrt{2}M_V^2} - \frac{\sqrt{2}F_V\kappa_{14}^V}{M_V^2} - \frac{F_V\kappa_{16}^V}{4\sqrt{2}M_V^2} \\
& + \frac{F_V^2\kappa_2^{\text{VV}}}{M_V^4} - \frac{F_V^2\kappa_3^{\text{VV}}}{8M_V^4} + \frac{\kappa^{\text{VVP}}d_mF_V^2}{2M_P^2M_V^4}, \\
C_8^W &= \frac{\mathbf{N_C}}{768M_0^2\pi^2} + \frac{F_V\kappa_{12}^V}{6\sqrt{2}M_V^2} - \frac{F_V\kappa_{13}^V}{\sqrt{2}M_V^2} + \frac{F_V\kappa_{16}^V}{12\sqrt{2}M_V^2} + \frac{F_V^2\kappa_1^{\text{VV}}}{2M_V^4} + \frac{F_V^2\kappa_3^{\text{VV}}}{24M_V^4} \\
& - \frac{\sqrt{2}d_{m0}F_V\kappa_3^{\text{PV}}}{3M_P^2M_V^2} + \frac{d_{m0}F_V^2\kappa^{\text{VVP}}}{6M_P^2M_V^4} + \frac{2d_{m0}\kappa_5^P}{3M_P^2}, \\
C_9^W &= -\frac{F_A\kappa_3^A}{4\sqrt{2}M_A^2} - \frac{F_A^2\kappa_3^{\text{AA}}}{8M_A^4} - \frac{F_A\kappa_8^A}{2\sqrt{2}M_A^2} - \frac{\sqrt{2}F_A\kappa_{11}^A}{M_A^2} - \frac{F_A\kappa_{12}^A}{\sqrt{2}M_A^2} - \frac{F_A\kappa_{15}^A}{4\sqrt{2}M_A^2}
\end{aligned}$$

$$\begin{aligned}
& + \frac{F_A^2 \kappa_2^{AA}}{M_A^4} + \frac{2d_m \kappa_1^P}{M_P^2} - \frac{\sqrt{2}d_m F_A \kappa_1^{PA}}{M_A^2 M_P^2} - \frac{d_m F_A \kappa_2^{PA}}{\sqrt{2}M_A^2 M_P^2} + \frac{\kappa^{AAP} d_m F_A^2}{2M_A^4 M_P^2}, \\
C_{10}^W &= \frac{\mathbf{N}_C}{768M_0^2 \pi^2} + \frac{F_A \kappa_3^A}{12\sqrt{2}M_A^2} + \frac{F_A^2 \kappa_3^{AA}}{24M_A^4} + \frac{F_A \kappa_8^A}{6\sqrt{2}M_A^2} - \frac{F_A \kappa_9^A}{\sqrt{2}M_A^2} + \frac{F_A \kappa_{10}^A}{2\sqrt{2}M_A^2} + \frac{F_A \kappa_{15}^A}{12\sqrt{2}M_A^2} \\
& + \frac{F_A^2 \kappa_1^{AA}}{2M_A^4} + \frac{F_A^2 \kappa^{AAP} d_{m0}}{6M_A^4 M_P^2} - \frac{\sqrt{2}F_A d_{m0} \kappa_1^{PA}}{3M_A^2 M_P^2} - \frac{F_A d_{m0} \kappa_2^{PA}}{3\sqrt{2}M_A^2 M_P^2} + \frac{2d_{m0} \kappa_1^P}{3M_P^2}, \\
C_{11}^W &= \frac{F_A \kappa_{14}^A}{\sqrt{2}M_A^2} + \frac{c_m \kappa_2^S}{M_S^2} - \frac{c_m F_A \kappa_1^{SA}}{\sqrt{2}M_A^2 M_S^2} + \frac{c_m F_V \kappa_1^{SV}}{\sqrt{2}M_S^2 M_V^2} + \frac{c_m F_V \kappa_2^{SV}}{2\sqrt{2}M_S^2 M_V^2} + \frac{F_V \kappa_4^V}{2\sqrt{2}M_V^2} - \frac{F_V \kappa_{15}^V}{\sqrt{2}M_V^2} \\
& + \frac{F_A F_V \kappa_6^{VA}}{2M_A^2 M_V^2} + \frac{\kappa^{VAS} c_m F_A F_V}{2M_A^2 M_S^2 M_V^2}, \\
C_{12}^W &= \frac{\sqrt{2}G_V \kappa_1^V}{M_V^2} - \frac{\sqrt{2}G_V \kappa_2^V}{M_V^2} - \frac{\sqrt{2}G_V \kappa_3^V}{M_V^2} + \frac{G_V^2 \kappa_3^{VV}}{M_V^4}, \\
C_{13}^W &= -\frac{F_A \kappa_3^A}{\sqrt{2}M_A^2} + \frac{F_A F_V \kappa_3^{VA}}{M_A^2 M_V^2} + \frac{F_V G_V \kappa_3^{VV}}{M_V^4} + \frac{2\sqrt{2}F_A \kappa_4^A}{M_A^2} + \frac{F_A \kappa_{15}^A}{\sqrt{2}M_A^2} - \frac{F_A^2 \kappa_4^{AA}}{2M_A^4} + \frac{4c_d \kappa_2^S}{M_S^2} \\
& - \frac{2\sqrt{2}c_d F_A \kappa_1^{SA}}{M_A^2 M_S^2} + \frac{2\sqrt{2}c_d F_V \kappa_1^{SV}}{M_S^2 M_V^2} + \frac{\sqrt{2}c_d F_V \kappa_2^{SV}}{M_S^2 M_V^2} + \frac{\sqrt{2}F_V \kappa_1^V}{M_V^2} - \frac{2\sqrt{2}F_V \kappa_5^V}{M_V^2} - \frac{\sqrt{2}F_V \kappa_6^V}{M_V^2} \\
& - \frac{\sqrt{2}F_V \kappa_8^V}{M_V^2} - \frac{\sqrt{2}G_V \kappa_{12}^V}{M_V^2} - \frac{G_V \kappa_{16}^V}{\sqrt{2}M_V^2} + \frac{\sqrt{2}G_V \kappa_{17}^V}{M_V^2} + \frac{2F_A F_V \kappa_1^{VA}}{M_A^2 M_V^2} + \frac{F_A F_V \kappa_2^{VA}}{M_A^2 M_V^2} \\
& + \frac{2\kappa^{VAS} c_d F_A F_V}{M_A^2 M_S^2 M_V^2}, \\
C_{14}^W &= -\frac{F_A \kappa_3^A}{2\sqrt{2}M_A^2} - \frac{F_V \kappa_3^V}{\sqrt{2}M_V^2} + \frac{F_A F_V \kappa_3^{VA}}{2M_A^2 M_V^2} + \frac{\sqrt{2}F_A \kappa_4^A}{M_A^2} + \frac{F_A \kappa_{15}^A}{2\sqrt{2}M_A^2} - \frac{F_A^2 \kappa_4^{AA}}{4M_A^4} + \frac{2c_d \kappa_2^S}{M_S^2} \\
& - \frac{\sqrt{2}c_d F_A \kappa_1^{SA}}{M_A^2 M_S^2} + \frac{\sqrt{2}c_d F_V \kappa_1^{SV}}{M_S^2 M_V^2} + \frac{c_d F_V \kappa_2^{SV}}{\sqrt{2}M_S^2 M_V^2} + \frac{F_V \kappa_1^V}{\sqrt{2}M_V^2} - \frac{\sqrt{2}F_V \kappa_5^V}{M_V^2} - \frac{F_V \kappa_6^V}{\sqrt{2}M_V^2} - \frac{F_V \kappa_8^V}{\sqrt{2}M_V^2} \\
& + \frac{F_A F_V \kappa_1^{VA}}{M_A^2 M_V^2} + \frac{F_A F_V \kappa_2^{VA}}{2M_A^2 M_V^2} + \frac{\kappa^{VAS} c_d F_A F_V}{M_A^2 M_S^2 M_V^2}, \\
C_{15}^W &= -\frac{F_A \kappa_3^A}{2\sqrt{2}M_A^2} + \frac{F_A F_V \kappa_3^{VA}}{2M_A^2 M_V^2} - \frac{F_V G_V \kappa_3^{VV}}{M_V^4} + \frac{\sqrt{2}F_A \kappa_4^A}{M_A^2} + \frac{F_A \kappa_{15}^A}{2\sqrt{2}M_A^2} - \frac{F_A^2 \kappa_4^{AA}}{4M_A^4} + \frac{2c_d \kappa_2^S}{M_S^2} \\
& - \frac{\sqrt{2}c_d F_A \kappa_1^{SA}}{M_A^2 M_S^2} + \frac{\sqrt{2}c_d F_V \kappa_1^{SV}}{M_S^2 M_V^2} + \frac{c_d F_V \kappa_2^{SV}}{\sqrt{2}M_S^2 M_V^2} + \frac{F_V \kappa_2^V}{\sqrt{2}M_V^2} - \frac{\sqrt{2}F_V \kappa_5^V}{M_V^2} - \frac{F_V \kappa_6^V}{\sqrt{2}M_V^2} - \frac{F_V \kappa_8^V}{\sqrt{2}M_V^2} \\
& - \frac{G_V \kappa_{17}^V}{\sqrt{2}M_V^2} + \frac{F_A F_V \kappa_1^{VA}}{M_A^2 M_V^2} + \frac{F_A F_V \kappa_2^{VA}}{2M_A^2 M_V^2} + \frac{\kappa^{VAS} c_d F_A F_V}{M_A^2 M_S^2 M_V^2}, \\
C_{16}^W &= \frac{\sqrt{2}F_A \kappa_2^A}{M_A^2} + \frac{F_A G_V \kappa_2^{VA}}{M_A^2 M_V^2} - \frac{\sqrt{2}G_V \kappa_6^V}{M_V^2} + \frac{\sqrt{2}G_V \kappa_7^V}{M_V^2} - \frac{\sqrt{2}G_V \kappa_8^V}{M_V^2} + \frac{F_A G_V \kappa_3^{VA}}{M_A^2 M_V^2} \\
& - \frac{F_A G_V \kappa_4^{VA}}{M_A^2 M_V^2} - \frac{G_V^2 \kappa_3^{VV}}{M_V^4}, \\
C_{17}^W &= \frac{F_A \kappa_1^A}{\sqrt{2}M_A^2} - \frac{c_d \kappa_1^S}{M_S^2} - \frac{\sqrt{2}c_d F_A \kappa_1^{SA}}{M_A^2 M_S^2} + \frac{\sqrt{2}c_d F_V \kappa_1^{SV}}{M_S^2 M_V^2} - \frac{\sqrt{2}c_d G_V \kappa_1^{SV}}{M_S^2 M_V^2} + \frac{F_A F_V \kappa_1^{VA}}{M_A^2 M_V^2}
\end{aligned}$$

$$\begin{aligned}
& -\frac{F_A G_V \kappa_1^{\text{VA}}}{M_A^2 M_V^2} + \frac{F_A \kappa_2^A}{\sqrt{2} M_A^2} + \frac{\sqrt{2} F_A \kappa_4^A}{M_A^2} + \frac{F_A \kappa_{15}^A}{2\sqrt{2} M_A^2} - \frac{F_A^2 \kappa_4^{\text{AA}}}{4M_A^4} + \frac{2c_d \kappa_2^S}{M_S^2} - \frac{c_d F_A \kappa_2^{\text{SA}}}{\sqrt{2} M_A^2 M_S^2} \\
& + \frac{c_d F_V \kappa_2^{\text{SV}}}{2\sqrt{2} M_S^2 M_V^2} - \frac{\sqrt{2} F_V \kappa_5^V}{M_V^2} + \frac{\sqrt{2} G_V \kappa_5^V}{M_V^2} - \frac{F_V \kappa_6^V}{\sqrt{2} M_V^2} - \frac{F_V \kappa_8^V}{\sqrt{2} M_V^2} + \frac{F_A F_V \kappa_2^{\text{VA}}}{2M_A^2 M_V^2} + \frac{F_A F_V \kappa_3^{\text{VA}}}{2M_A^2 M_V^2} \\
& - \frac{F_V G_V \kappa_3^{\text{VV}}}{4M_V^4} + \frac{G_V^2 \kappa_4^{\text{VV}}}{2M_V^4} - \frac{F_V G_V \kappa_4^{\text{VV}}}{4M_V^4} + \frac{\kappa^{\text{VAS}} c_d F_A F_V}{M_A^2 M_S^2 M_V^2} - \frac{\kappa^{\text{VAS}} c_d F_A G_V}{M_A^2 M_S^2 M_V^2}, \\
C_{18}^W &= 0, \\
C_{19}^W &= -\frac{\sqrt{2} F_A \kappa_5^A}{M_A^2} - \frac{F_A G_V \kappa_5^{\text{VA}}}{M_A^2 M_V^2} - \frac{\sqrt{2} F_V \kappa_6^V}{M_V^2} - \frac{\sqrt{2} G_V \kappa_{11}^V}{M_V^2} + \frac{F_V \kappa_{16}^V}{\sqrt{2} M_V^2} - \frac{G_V \kappa_{16}^V}{\sqrt{2} M_V^2} + \frac{\sqrt{2} G_V \kappa_{17}^V}{M_V^2} \\
& + \frac{F_A F_V \kappa_3^{\text{VA}}}{M_A^2 M_V^2} + \frac{F_V G_V \kappa_3^{\text{VV}}}{M_V^4} - \frac{F_V^2 \kappa_4^{\text{VV}}}{2M_V^4} + \frac{F_V G_V \kappa_4^{\text{VV}}}{M_V^4}, \\
C_{20}^W &= \frac{\sqrt{2} F_A \kappa_4^A}{M_A^2} - \frac{F_A^2 \kappa_4^{\text{AA}}}{2M_A^4} - \frac{F_V^2 \kappa_4^{\text{VV}}}{4M_V^4} + \frac{F_V G_V \kappa_4^{\text{VV}}}{2M_V^4} - \frac{F_A \kappa_5^A}{\sqrt{2} M_A^2} - \frac{F_A \kappa_6^A}{\sqrt{2} M_A^2} + \frac{F_A \kappa_{15}^A}{\sqrt{2} M_A^2} + \frac{2c_d \kappa_2^S}{M_S^2} \\
& - \frac{\sqrt{2} c_d F_A \kappa_1^{\text{SA}}}{M_A^2 M_S^2} + \frac{\sqrt{2} c_d F_V \kappa_1^{\text{SV}}}{M_S^2 M_V^2} + \frac{c_d F_V \kappa_2^{\text{SV}}}{\sqrt{2} M_S^2 M_V^2} - \frac{\sqrt{2} F_V \kappa_5^V}{M_V^2} - \frac{\sqrt{2} F_V \kappa_6^V}{M_V^2} - \frac{\sqrt{2} F_V \kappa_8^V}{M_V^2} \\
& + \frac{F_V \kappa_{16}^V}{2\sqrt{2} M_V^2} - \frac{G_V \kappa_{16}^V}{\sqrt{2} M_V^2} + \frac{F_A F_V \kappa_1^{\text{VA}}}{M_A^2 M_V^2} + \frac{F_A F_V \kappa_2^{\text{VA}}}{M_A^2 M_V^2} + \frac{F_A F_V \kappa_3^{\text{VA}}}{M_A^2 M_V^2} - \frac{F_V G_V \kappa_3^{\text{VV}}}{M_V^4} \\
& + \frac{\kappa^{\text{VAS}} c_d F_A F_V}{M_A^2 M_S^2 M_V^2}, \\
C_{21}^W &= -\frac{\sqrt{2} F_A \kappa_4^A}{M_A^2} + \frac{F_A^2 \kappa_4^{\text{AA}}}{2M_A^4} - \frac{F_A F_V \kappa_4^{\text{VA}}}{2M_A^2 M_V^2} + \frac{F_V^2 \kappa_4^{\text{VV}}}{4M_V^4} - \frac{F_V G_V \kappa_4^{\text{VV}}}{2M_V^4} + \frac{F_A \kappa_7^A}{\sqrt{2} M_A^2} - \frac{F_A \kappa_{15}^A}{\sqrt{2} M_A^2} \\
& - \frac{2c_d \kappa_2^S}{M_S^2} + \frac{\sqrt{2} c_d F_A \kappa_1^{\text{SA}}}{M_A^2 M_S^2} - \frac{\sqrt{2} c_d F_V \kappa_1^{\text{SV}}}{M_S^2 M_V^2} - \frac{c_d F_V \kappa_2^{\text{SV}}}{\sqrt{2} M_S^2 M_V^2} + \frac{\sqrt{2} F_V \kappa_5^V}{M_V^2} + \frac{F_V \kappa_6^V}{\sqrt{2} M_V^2} + \frac{F_V \kappa_7^V}{\sqrt{2} M_V^2} \\
& + \frac{F_V \kappa_8^V}{\sqrt{2} M_V^2} - \frac{F_V \kappa_{16}^V}{2\sqrt{2} M_V^2} + \frac{G_V \kappa_{16}^V}{\sqrt{2} M_V^2} - \frac{G_V \kappa_{17}^V}{\sqrt{2} M_V^2} - \frac{F_A F_V \kappa_1^{\text{VA}}}{M_A^2 M_V^2} - \frac{F_A F_V \kappa_2^{\text{VA}}}{2M_A^2 M_V^2} - \frac{F_A F_V \kappa_3^{\text{VA}}}{2M_A^2 M_V^2} \\
& - \frac{F_V G_V \kappa_3^{\text{VV}}}{2M_V^4} - \frac{\kappa^{\text{VAS}} c_d F_A F_V}{M_A^2 M_S^2 M_V^2}, \\
C_{22}^W &= -\frac{F_V \kappa_{17}^V}{\sqrt{2} M_V^2} - \frac{F_V^2 \kappa_3^{\text{VV}}}{2M_V^4}, \\
C_{23}^W &= -\frac{F_A \kappa_{16}^A}{\sqrt{2} M_A^2} - \frac{F_A^2 \kappa_3^{\text{AA}}}{2M_A^4}. \tag{91}
\end{aligned}$$

Apart from already mentioned relations $C_3^W = 0$ and $C_{18}^W = 0$ we have found out one further relation free from κ_i^X

$$\frac{F_V^2}{2G_V} C_{12}^W = F_V (C_{14}^W - C_{15}^W) + G_V C_{22}^W. \tag{92}$$

The transformation established in Appendix B was employed as an independent check of the previous relations.

6 Summary

In this paper we have studied the odd-intrinsic sector of the low-energy QCD. We have constructed the most general resonance Lagrangian that describes the interactions of the Goldstone bosons and the lowest-lying vector-, axial-, scalar-, pseudoscalar-resonance multiplets. We were working in the large N_C approximation and considered only those terms that contribute to $O(p^6)$ anomalous Lagrangian (i.e. to the first non-trivial order). This was the main aim of our work. We then demonstrated the use of this Lagrangian for three different applications. The first two represent calculations of two three-point Green functions $\langle VVP \rangle$ and $\langle VAS \rangle$. The third application was the complete integration out of the resonance fields and establishing the so-called saturation of LECs by resonance fields.

The first application VVP is the most important example of the odd-intrinsic sector, both from the theoretical and phenomenological point of view. We have discussed different aspects of this Green function. First, after calculating this three-point correlator within our model and imposing a certain high-energy constraint we ended up with the result which depends only on two parameters. These were further set using new BABAR data on $\pi\gamma\gamma$ off-shell formfactor and Belle collaboration's limit on $\pi' \rightarrow \gamma\gamma$ decay. After setting these two parameters we can make further predictions. The outcome of our analysis is for example a very precise determination of the decay width of a process $\rho \rightarrow \pi\gamma$: $\Gamma_{\rho \rightarrow \pi\gamma} = 67(2.3)$ keV. We have also studied a relative dependence of the rare decays $\pi' \rightarrow \gamma\gamma$ and $\pi' \rightarrow \rho\gamma$. Based on the experimental upper limit of the former one can set the lowest limit of the latter. Prediction of our model is $30 \text{ keV} \gtrsim \Gamma_{\pi' \rightarrow \rho\gamma} \gtrsim 4 \text{ keV}$ (based on Belle's $\Gamma_{\pi' \rightarrow \gamma\gamma} \lesssim 72 \text{ eV}$). Next, we have also evaluated the value of C_7^W LEC together with short discussion on π^0 and η two photon decays. Last but not least a very precise determination of the off-shell π^0 -pole contribution to the muon $g - 2$ factor was provided. Our final determination of this factor is $a_\mu^{\pi^0} = 65.8(1.2) \times 10^{-11}$. The $R\chi T$ approach has thus reduced the error of the similar determination based on lowest-meson saturation ansatz by factor of ten and is in exact agreement with the most recent determination based on AdS/QCD assumptions. Let us note that the present theoretical error for the complete anomalous magnetic moment of the muon is around 50×10^{-11} and the experimental error around 60×10^{-11} [61] (with the well-known discrepancy above 3σ). A new proposed experiment at Fermilab E989 [62] plans to go down with the precision to the preliminary value 16×10^{-11} and thus the reduction of the error in the theoretical light-by-light calculation is more than desirable.

If VVP represents very important and rich phenomenological example, the three-point correlator $\langle VAS \rangle$ is connected with very rare processes and represents so far never studied example of the odd-sector. We have established its OPE behaviour which enabled us to reduce the dependence of the VAS Green function to one parameter. This opens the possibility of a future study of these rare but interesting processes.

In the last section we have studied the resonance saturation at low energies. We have integrated out the resonance fields to establish the dependence of LECs of odd-sector C_i^W on our parameters. As we are limited by large N_C we cannot make prediction for C_3^W and C_{18}^W but we have set all other 21 LECs. We have found one relation among C_{12}^W , C_{14}^W , C_{15}^W and C_{22}^W free from our parameters.

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A The large N_C counting

A.1 General considerations

Let us start with the $U_L(N_F) \times U_R(N_F)$ invariant Lagrangian for the nonet of the GB and resonances without using the equations of motion and the Cayley-Hamilton identities. Then the large N_C behaviour of the couplings accompanying individual operators in the effective Lagrangian with octet GB (after η' has been integrated out) can be understood as follows.

Let us write in the same way as in [24]

$$\tilde{u} = e^{i\phi^0 T^0/F\sqrt{2}} u, \quad (93)$$

where $T^0 = \sqrt{1/N_F} \mathbf{1}$ and

$$u = e^{i\phi^a T^a/F\sqrt{2}}$$

is the $SU_L(N_F) \times SU_R(N_F)$ basic building block, and therefore

$$\phi^0 = \frac{F}{i} \sqrt{\frac{2}{N_F}} \ln(\det \tilde{u}). \quad (94)$$

Let us also remind [24], that ϕ^0 and ϕ^a do not mix under the nonlinearly realized $U_L(N_F) \times U_R(N_F)$ symmetry. For the construction of the $U_L(N_F) \times U_R(N_F)$ effective Lagrangian, we have the usual building blocks constructed from \tilde{u} and the usual external sources l_μ, r_μ, χ and χ^+ (now also with singlet components) *e.g.*

$$\begin{aligned} \tilde{u}_\mu &= u_\mu - D_\mu \phi^0 \frac{\sqrt{2} T^0}{F}, \\ \tilde{\chi}_\pm &= e^{-i\phi^0 \sqrt{2} T^0/F} u^+ \chi u^\pm \pm e^{i\phi^0 \sqrt{2} T^0/F} u \chi^\pm u \\ &= \chi_\pm - \frac{i}{F} \sqrt{\frac{2}{N_F}} \phi^0 \chi_\mp + \dots, \\ \langle l_\mu \rangle &= l_\mu^0 \sqrt{\frac{N_F}{2}}, \end{aligned} \quad (95)$$

etc. at our disposal. In the above formulae, the covariant (in fact invariant) derivative of ϕ^0 is defined as

$$D_\mu \phi^0 = \partial_\mu \phi^0 - 2a_\mu^0 F, \quad (96)$$

however, it does not represent an independent building block because of the identity

$$\langle \tilde{u}_\mu \rangle = \sqrt{\frac{N_F}{2}} \frac{D_\mu \phi^0}{F}. \quad (97)$$

The above set of building block have to be further enlarged including also the external sources θ for the winding number density

$$\omega = \frac{g^2}{16\pi^2} \text{tr}_c G_{\mu\nu} \tilde{G}^{\mu\nu}, \quad (98)$$

with covariant derivative

$$D_\mu \theta = \partial_\mu \theta + 2a_\mu^0.$$

We have to include also the following invariant combination

$$X = \theta + \frac{\phi^0}{F}. \quad (99)$$

Let us remind the large N_C counting for the generating functional of the connected Green function of quark bilinears and winding number densities

$$Z[l, r, \chi, \chi^+, \theta] = N_C^2 Z_0[\theta/N_C] + N_C Z_1[l, r, \chi, \chi^+, \theta/N_C] + \dots, \quad (100)$$

where the ellipses stay for the subleading terms in the $1/N_C$ expansion. This implies the usual N_C counting of the physical amplitudes with g glueballs and m mesons

$$\mathcal{A}_{g,m} = O(N_C^{1+\delta_{m0}-g-\frac{m}{2}}). \quad (101)$$

This counting should be reflected within the construction of the effective chiral Lagrangian of $R\chi T$.

According to the (101), the explicit resonance fields have to be counted as $O(N_C^{-1/2})$. As far as the GB are concerned, within the tilded building blocs, each member of the pseudoscalar nonet is automatically accompanied with (minus)one power of the decay constant $F = O(N_C^{1/2})$, which ensures the right counting of the vertices with GB, provided the corresponding fields are counted as $O(N_C^0)$. The only subtlety is connected with the field ϕ^0 .

The origin of the field ϕ^0 in the individual terms of the Lagrangian is twofold. It can either come from the tilded building blocks $Y = \tilde{u}_\mu, \tilde{h}_{\mu\nu}, \tilde{\chi}_\pm$ (and from their covariant derivatives $\tilde{D}_\mu Y$; note that it completely decouples from Γ_μ and $f_\pm^{\mu\nu}$) or from the X -dependence of the Lagrangian. Each operator $\tilde{\mathcal{O}}$ constructed from the tilded building blocks only (and therefore including at least one flavour trace, the only exception is $\tilde{\mathcal{O}} = 1$) is in general accompanied by a potential $V_{\tilde{\mathcal{O}}}(X)$ which is a function of the variable X only,

$$\tilde{\mathcal{L}} = \sum_{\tilde{\mathcal{O}}} V_{\tilde{\mathcal{O}}}(X) \tilde{\mathcal{O}}. \quad (102)$$

While ϕ^0 originating from the tilded operators is counted as $O(N_C^0)$ as the other GB, however, the same field coming from the power expansion of the potentials counts as $O(1/N_C)$ within the large N_C expansion. Therefore, expanding the general operator $\tilde{\mathcal{O}}$ and the corresponding potential $V_{\tilde{\mathcal{O}}}(X)$ in powers of ϕ^0 and its derivatives (and taking into account that $F = O(N_C^{1/2})$) we have the following natural rule for the order $O(N_C^n)$ of the resulting coupling constant at a term of this expansion with T flavour traces, R resonance fields and n_0 fields ϕ^0

$$2 - T - \frac{1}{2}R - \frac{3}{2}n_0 \leq n \leq 2 - T - \frac{1}{2}R - \frac{1}{2}n_0. \quad (103)$$

The lower or higher bounds are saturated in the case when all ϕ^0 's come exclusively either from $V_{\tilde{\mathcal{O}}}(X)$ or from $\tilde{\mathcal{O}}$.

Suppose that we had used the LO GB equations of motion prior to the expansion in powers of ϕ^0 . This allows to eliminate the terms with derivatives, namely [24]

$$\nabla^\mu \tilde{u}_\mu = \tilde{\chi}_- + \frac{4}{\sqrt{2N_F}} M_0^2 \frac{\phi^0}{F}. \quad (104)$$

Such a transformation of the original tilded operator do not create any extra trace in contrast to the octet case. Because the singlet mass $M_0^2 = O(1/N_C)$, the ϕ^0 dependence of the resulting operator brings about a factor of the order $O(N_C^{-3/2})$ (the same, as if ϕ^0 came from the potential) and the above bounds on n remain therefore valid. On the other hand, the further simplification using the Cayley-Hamilton identity can destroy them, provided we use it in order to eliminate terms with less traces in favour of the terms with more traces.

The next step is to integrate out ϕ^0 treating the mass M_0^2 as $O(p^0)$. This can be done using its equation of motion, derived from the corresponding part of the LO Lagrangian expanded in powers of ϕ^0

$$\mathcal{L}_0^{(2)} = \frac{1}{2} D\phi^0 \cdot D\phi^0 - \frac{1}{2} M_0^2 (\phi^0)^2 - i \frac{F}{2\sqrt{2N_F}} \langle \chi_- \rangle \phi^0 + d_0 \langle P \rangle \phi^0 + \dots, \quad (105)$$

where d_0 term comes from the expansion of the potential and is therefore of the order $O(N_C^{-1})$. The solution for ϕ^0 reads in the leading order of the p expansion⁴

$$\begin{aligned} \phi^{0(2)} &= \frac{1}{M_0^2} \left(i \frac{F}{2\sqrt{2N_F}} \langle \chi_- \rangle + d_0 \langle P \rangle \right) \\ &= O(N_C^{3/2}) + O(N_C^0), \end{aligned} \quad (106)$$

where we have depicted the orders of both terms. $\phi^{0(2)}$ should then be inserted into the original Lagrangian expanded in powers of ϕ^0 . As a result, taking (103) into account, the orders of the multiple trace operators within the $SU_L(N_F) \times SU_R(N_F)$ operator basis are enhanced. Namely, we have the following bound for the corresponding couplings

$$2 - T_0 - \frac{1}{2} R_0 - \frac{3}{2} n_{\langle P \rangle} \leq n \leq 2 - T_0 - \frac{1}{2} R_0 + n_{\langle \chi_- \rangle} - \frac{1}{2} n_{\langle P \rangle}, \quad (107)$$

where T_0 and R_0 are the numbers of the traces and resonance fields before elimination of ϕ^0 and $n_{\langle \chi_- \rangle}$ and $n_{\langle P \rangle}$ are the numbers of the new factors $\langle \chi_- \rangle$ and $\langle P \rangle$ (which appear after ϕ^0 is integrated out) respectively. More conveniently this can be expressed in terms of the actual number of traces $T = T_0 + n_{\langle P \rangle} + n_{\langle \chi_- \rangle}$ and resonances $R = R_0 + n_{\langle P \rangle}$ as

$$2 - T - \frac{1}{2} R + n_{\langle \chi_- \rangle} \leq n \leq 2 - T - \frac{1}{2} R + n_{\langle P \rangle} + 2n_{\langle \chi_- \rangle}. \quad (108)$$

The loophole of this formula is, that for its application one has to trace back which of the factors $\langle P \rangle$ and $\langle \chi_- \rangle$ originate in the ϕ^0 dependence of the tilded Lagrangian. The extreme cases are either none or all of them, which gives a much raw estimate

$$2 - T - \frac{1}{2} R \leq n \leq 2 - T - \frac{1}{2} R + N_{\langle P \rangle} + 2N_{\langle \chi_- \rangle}, \quad (109)$$

where now $N_{\langle \chi_- \rangle}$ and $N_{\langle P \rangle}$ are the total numbers of $\langle P \rangle$ and $\langle \chi_- \rangle$ traces in the operator, the lower bound corresponds now to the usual trace and resonance counting.

⁴Here we have took into account, that the resonance fields should be counted as $O(p^2)$.

A.2 Explicit examples

Let us illustrate the above statements by means of an explicit examples. For instance, the coupling at the term $\langle S\chi_- \rangle \langle \chi_- \rangle$, at first sight of the order $O(N_C^{-1/2})$ might be of the order $O(N_C^{1/2})$ or even $O(N_C^{3/2})$, because it can originate either from the term

$$\begin{aligned} i\langle S\tilde{\chi}_- \rangle W_{\langle S\chi_- \rangle}(X) &= i\langle S(\chi_- + \dots) \rangle \left(w_{\langle S\chi_- \rangle}^1 X + \dots \right) \\ &\rightarrow -\langle S\chi_- \rangle \frac{1}{M_0^2} \left(w_{\langle S\chi_- \rangle}^1 \frac{1}{2\sqrt{2N_F}} \langle \chi_- \rangle + \dots \right), \end{aligned} \quad (110)$$

which has the constant $w_{\langle S\chi_- \rangle}^1 = O(N_C^{-1/2})$, (this corresponds to the lower bound (108)) or from the term

$$\begin{aligned} \langle S\tilde{\chi}_+ \rangle W_{\langle S\chi_+ \rangle}(X) &= \langle S \left(\chi_+ - \frac{i}{F} \sqrt{\frac{2}{N_F}} \phi^0 \chi_- + \dots \right) \rangle \left(w_{\langle S\chi_+ \rangle}^0 + \dots \right) \\ &= -w_{\langle S\chi_+ \rangle}^0 \frac{i}{F} \sqrt{\frac{2}{N_F}} \phi^0 \langle S\chi_- \rangle + \dots \\ &\rightarrow \frac{1}{M_0^2} w_{\langle S\chi_+ \rangle}^0 \left(\frac{1}{2N_F} \right) \langle S\chi_- \rangle \langle \chi_- \rangle + \dots, \end{aligned} \quad (111)$$

where $w_{\langle S\chi_+ \rangle}^0 = O(N_C^{1/2})$; (this corresponds to the upper bound (108)).

Similarly the coupling d_{m0} at the operator $i\langle P \rangle \langle \chi_- \rangle$ (see (14)), naively of the order $O(N_C^{-1/2})$ can be enhanced by the ϕ^0 exchange. Indeed, inserting (106) to the term $d_0 \langle P \rangle \phi^0$ of the Lagrangian (105), we get the following contribution to d_{m0}

$$d_{m0} = \frac{d_0}{M_0^2} \frac{F\sqrt{N_F}}{2\sqrt{2}} = O(N_C^{1/2}), \quad (112)$$

where we have taken into account that $d_0 = O(N_C^{-1})$.

Let us give also some examples of the odd intrinsic parity terms with resonances, which similarly to the previous example lead to N_C enhanced multiple trace terms when ϕ^0 is integrated out. Some terms with one resonance are for example

$$\tilde{\mathcal{L}}_R = \varepsilon_{\mu\nu\alpha\beta} \langle V^{\mu\nu} [\tilde{u}^\alpha, \tilde{u}^\beta] \rangle W_{R1}(X) + \varepsilon_{\mu\nu\alpha\beta} \langle V^{\mu\nu} f_+^{\alpha\beta} \rangle W_{R2}(X) + \varepsilon_{\mu\nu\alpha\beta} \langle A^{\mu\nu} f_+^{\alpha\beta} \rangle W_{R3}(X), \quad (113)$$

where

$$W_{Ri}(X) = \sum_k w_{Ri}^{(k)} X^k \quad (114)$$

and where

$$w_{Ri}^{(0)} = 0, \quad w_{Ri}^{(1)} = O(N_C^{-1/2}), \quad \text{for } i = 1, 2, 3. \quad (115)$$

These generate the operators

$$\begin{aligned} \hat{\mathcal{O}}_{18}^V &= \varepsilon_{\mu\nu\alpha\beta} \langle V^{\mu\nu} [u^\alpha, u^\beta] \rangle \langle \chi_- \rangle, \\ \hat{\mathcal{O}}_{13}^V &= i\varepsilon_{\mu\nu\alpha\beta} \langle V^{\mu\nu} f_+^{\alpha\beta} \rangle \langle \chi_- \rangle, \end{aligned}$$

$$\hat{\mathcal{O}}_9^A = i\langle A^{\mu\nu} f_+^{\alpha\beta} \rangle \langle \chi_- \rangle \quad (116)$$

with the couplings of the order $O(N_C^{1/2})$ (*i.e.* of the same order as analogous single trace operators and therefore included in our basis) and

$$\begin{aligned} & \varepsilon_{\mu\nu\alpha\beta} \langle V^{\mu\nu} [u^\alpha, u^\beta] \rangle \langle P \rangle, \\ & \varepsilon_{\mu\nu\alpha\beta} \langle V^{\mu\nu} f_+^{\alpha\beta} \rangle \langle P \rangle, \\ & \langle A^{\mu\nu} f_+^{\alpha\beta} \rangle \langle P \rangle \end{aligned} \quad (117)$$

with the couplings of the order $O(N_C^{-1})$ suppressed with respect to the single trace operators. The two-resonance example is

$$\tilde{\mathcal{L}}_{RR} = \varepsilon_{\mu\nu\alpha\beta} \langle V^{\mu\nu} V^{\alpha\beta} \rangle W_{RR1}(X) + \varepsilon_{\mu\nu\alpha\beta} \langle A^{\mu\nu} A^{\alpha\beta} \rangle W_{RR1}(X), \quad (118)$$

where

$$W_{RRi}(X) = \sum_k w_{RRi}^{(k)} X^k \quad \text{with} \quad w_{RRi}^{(0)} = 0, \quad w_{RRi}^{(1)} = O(N_C^{-1}), \quad \text{for} \quad i = 1, 2.$$

It gives rise to the operators

$$\begin{aligned} \hat{\mathcal{O}}_1^{VV} &= i\varepsilon_{\mu\nu\alpha\beta} \langle V^{\mu\nu} V^{\alpha\beta} \rangle \langle \chi_- \rangle, \\ \hat{\mathcal{O}}_1^{AA} &= i\varepsilon_{\mu\nu\alpha\beta} \langle A^{\mu\nu} A^{\alpha\beta} \rangle \langle \chi_- \rangle \end{aligned} \quad (119)$$

with the couplings of the order $O(N_C^0)$ (the same order as the analogous single trace operators and therefore included in our basis) and $O(N_C^{-3/2})$ operators

$$\begin{aligned} & \varepsilon_{\mu\nu\alpha\beta} \langle V^{\mu\nu} V^{\alpha\beta} \rangle \langle P \rangle, \\ & \varepsilon_{\mu\nu\alpha\beta} \langle A^{\mu\nu} A^{\alpha\beta} \rangle \langle P \rangle, \end{aligned} \quad (120)$$

which are suppressed with respect to the single trace ones.

As the last step, we integrate out the resonance fields in order to get the resonance contribution to the odd parity sector LECs of the resulting χPT Lagrangian. It can be done using the $O(p^2)$ EOM for the resonance fields and inserting their solution $R^{(2)}$ back to the $R\chi T$ Lagrangian. The general form reads

$$R^{(2)} = \frac{1}{M_R^2} J_R^{(2)}, \quad (121)$$

where $J_R^{(2)} = O(p^2)$ comes from the LO resonance Lagrangian (14). Because $J_R^{(2)} = O(N_C^{1/2})$, the order of the contribution of the individual terms of the $R\chi T$ Lagrangian (with ϕ^0 integrated out) can be obtained counting the resonance fields as $O(N_C^{1/2})$. This gives finally the following simple bound on the order of the contribution of the operator with T traces, total $N_{\langle P \rangle}$ factors $\langle P \rangle$ and total $N_{\langle \chi_- \rangle}$ factors $\langle \chi_- \rangle$ originating in the to the LECs

$$2 - T \leq n \leq 2 - T + N_{\langle P \rangle} + 2N_{\langle \chi_- \rangle}. \quad (122)$$

The lower bound represents the usual trace counting. Note however, that the upper bound have to be taken with some caution, because it can be saturated only in the case when all $\langle P \rangle$

and $\langle \chi_- \rangle$ traces appear as a consequence of the ϕ^0 dependence and that this ϕ^0 dependence comes solely from the tilded operators and not from the potentials. For a given operator these two conditions need not to be satisfied simultaneously.

The fact that the N_C of order some operators can be enhanced could further complicate the usual way of the saturation of the ChPT LECs. Namely, in the process of integrating out the resonances, it is assumed, that loops can give only NLO contribution suppressed by the factor $1/N_C$ for each loop. This counting could be apparently complicated by the enhanced operators. Let us illustrate this point assuming the contribution of the following term of the odd R χ T Lagrangian $\tilde{\mathcal{L}}$

$$\begin{aligned}\tilde{\mathcal{L}} &= \dots + W_2^{AP}(X) \varepsilon_{\mu\nu\alpha\beta} \langle \{A^{\mu\nu}, \nabla^\alpha P\} \tilde{u}^\beta \rangle + \dots \\ &= \dots - 2w_2^{AP} \varepsilon_{\mu\nu\alpha\beta} \langle A^{\mu\nu} \nabla^\alpha P \rangle \sqrt{\frac{2}{N_F}} \frac{D^\beta \phi^0}{F} + \dots\end{aligned}\quad (123)$$

with $w_2^{AP} = O(N_C^0)$. This gives rise to the following enhanced N_C term

$$- \frac{i}{N_F} w_2^{AP} \frac{1}{M_0^2} \varepsilon_{\mu\nu\alpha\beta} \langle A^{\mu\nu} \nabla^\alpha P \rangle \partial^\beta \langle \chi_- \rangle = O(N_C). \quad (124)$$

Apparently, this term contributes to the $O(p^8)$ LECs, when the resonances are integrated out at the *tree level*. However, the bubble with two such vertices gives a contribution to the $O(p^6)$ operator $\partial_\alpha \langle \chi_- \rangle \partial^\alpha \langle \chi_- \rangle$ of the enhanced order $O(N_C^2)$. The same is true also for analogous operators from the even sector, *e.g.*

$$V_1^{SP}(X) \langle \{D_\mu S, P\} \tilde{u}^\mu \rangle = 2v_1^{SP} \langle P D_\mu S \rangle \sqrt{\frac{2}{N_F}} \frac{D^\mu \phi^0}{F} + \dots \quad (125)$$

with $v_1^{SP} = O(N_C^0)$ which leads to the enhanced operator

$$\frac{i}{N_F} v_1^{SP} \frac{1}{M_0^2} \langle P \nabla_\mu S \rangle \partial^\mu \langle \chi_- \rangle = O(N_C) \quad (126)$$

counted as $O(p^8)$ in the tree level saturation process. The bubble with two vertices

$$\frac{i}{N_F} v_1^{SP} \frac{1}{M_0^2} \langle P \partial_\mu S \rangle \partial^\mu \langle \chi_- \rangle \quad (127)$$

leads to the expression

$$\begin{aligned}& \frac{N_F^2}{2} \left(\frac{i}{N_F} v_1^{SP} \frac{1}{M_0^2} \right)^2 \int d^d x d^d y \partial^\mu \langle \chi_-(x) \rangle \partial^\nu \langle \chi_-(y) \rangle \\ & \times \int \frac{d^d k}{(2\pi)^d} e^{ik \cdot (x-y)} \frac{d^d p}{(2\pi)^d} \frac{p_\mu p_\nu}{(p^2 - M_S^2 + i0) ((p-k)^2 - M_P^2 + i0)} \\ & = \frac{i}{2} \left(\frac{v_1^{SP}}{M_0^2} \right)^2 \int d^d x \partial^\mu \langle \chi_-(x) \rangle \partial_\mu \langle \chi_-(x) \rangle \frac{(M_P^2)^{2-\varepsilon} - (M_S^2)^{2-\varepsilon}}{M_P^2 - M_S^2} \frac{1}{32\pi^2} \Gamma(\varepsilon - 2) (4\pi)^\varepsilon \\ & + O(p^8)\end{aligned}\quad (128)$$

and (after addition of appropriate counterterm) results in the following $O(N_C^2)$ contribution to the coupling $C_{\partial_\alpha \langle \chi_- \rangle \partial^\alpha \langle \chi_- \rangle}$ associated with $O(p^6)$ operator $\partial_\alpha \langle \chi_- \rangle \partial^\alpha \langle \chi_- \rangle$

$$C_{\partial_\alpha \langle \chi_- \rangle \partial^\alpha \langle \chi_- \rangle}^{PS-loop} = - \frac{1}{64\pi^2} \left(\frac{v_1^{SP}}{M_0^2} \right)^2 \frac{M_P^4 \left(\ln \frac{M_P^2}{\mu^2} + \gamma - \frac{1}{2} \right) - M_S^4 \left(\ln \frac{M_S^2}{\mu^2} + \gamma - \frac{1}{2} \right)}{M_P^2 - M_S^2}. \quad (129)$$

Though the above loop contribution are enhanced by the factor N_C^2 with respect to the naive trace counting, it does not mean, that loop counting fails. The reason is that the LO contribution to $C_{\partial_\alpha \langle \chi_- \rangle \partial^\alpha \langle \chi_- \rangle}$ that comes from the tree level and originates in the kinetic term of the field ϕ^0

$$\frac{1}{2} \partial_\mu \phi^0 \cdot \partial^\mu \phi^0 \rightarrow -\frac{1}{2} \left(\frac{F}{2M_0^2 \sqrt{2N_F}} \right)^2 \partial_\alpha \langle \chi_- \rangle \partial^\alpha \langle \chi_- \rangle = O(N_C^3) \quad (130)$$

so that the loop contribution is suppressed by $1/N_C$ as usual.

Let us finally comment briefly on one point, which also might lead to confusion. In [24], the following operators are abandon using the large N_C arguments, namely

$$\begin{aligned} & i \langle P u_\mu u^\mu \rangle \langle \chi_- \rangle, \\ & i \langle SP \rangle \langle \chi_- \rangle, \\ & i \langle \nabla_\mu \nabla^\mu \chi_- \rangle \langle P \rangle. \end{aligned} \quad (131)$$

These can be, however, derived from the operators (before doing any transformations)

$$\begin{aligned} & i \langle P \tilde{u}_\mu \tilde{u}^\mu \rangle V(X), \\ & i \langle SP \rangle V(X), \\ & i \langle \nabla_\mu \nabla^\mu \tilde{\chi}_+ \rangle V(X), \end{aligned} \quad (132)$$

by means of integrating out the field ϕ^0 , which appears from the potential for the first two operators (and saturates therefore the lower bound of (108)) and from the building block $\tilde{\chi}_+$ for the last one (and corresponds therefore to the upper bound of (108)). According to our rules the operators are of the order $O(N_C^{1/2})$, $O(N_C^0)$ and $O(N_C^{1/2})$ respectively (as the similar operators without additional trace) and all of them contribute therefore at the $O(N_C)$ order of the LECs of the effective chiral Lagrangian staying at the operators

$$\begin{aligned} & \langle \chi_- u_\mu u^\mu \rangle \langle \chi_- \rangle, \\ & \langle \chi_+ \chi_- \rangle \langle \chi_- \rangle, \\ & \langle \nabla_\mu \nabla^\mu \chi_- \rangle \langle \chi_- \rangle. \end{aligned} \quad (133)$$

However, these operators can be derived analogously as above from

$$\begin{aligned} & \langle \tilde{\chi}_+ \tilde{u}_\mu \tilde{u}^\mu \rangle, \\ & \langle \tilde{\chi}_+ \tilde{\chi}_+ \rangle, \\ & \langle \nabla_\mu \nabla^\mu \tilde{\chi}_+ \rangle, \end{aligned} \quad (134)$$

by the process which saturates the upper bound of (108) and results in the order $O(N_C^2)$. The abandoned operators lead therefore to the NLO contribution to the corresponding LECs.

B Field redefinition

As we have discussed in detail in Section 3, by means of appropriate field redefinition we can effectively eliminate subset of the $O(p^6)$ operators from the Lagrangian $\mathcal{L}_{R\chi T}^{(6, \text{ odd})}$ and shift their influence on the ChPT LECs into the effective coefficients $\overline{\kappa_i^X}$ which stay at the remaining

operators of the chiral order $O(p^6)$ and higher. As a consequence, the $O(p^6)$ LECs resulting from the process of integrating out the resonance fields from the Lagrangian $\mathcal{L}_{R\chi T}$ depend only on these effective couplings $\overline{\kappa}_i^X$ which are particular linear combinations of the original resonance couplings κ_i^X . In order to identify these relevant combinations and the redundant operators, we can proceed in several steps.

B.1 Elimination of $\mathcal{O}_{1,2}^{VV}$, $\mathcal{O}_{1,2}^{AA}$, \mathcal{O}^{VVP} and \mathcal{O}^{AAP}

With the field redefinitions

$$\begin{aligned} V_{\mu\nu} &\rightarrow V_{\mu\nu} - \frac{2}{M_V^2} \varepsilon_{\mu\nu\alpha\beta} \left(i\kappa_1^{VV} \langle \chi_- \rangle V^{\alpha\beta} + i\kappa_2^{VV} \{ \chi_-, V^{\alpha\beta} \} + \frac{1}{2} \kappa^{VVP} \{ P, V^{\alpha\beta} \} \right), \\ A_{\mu\nu} &\rightarrow A_{\mu\nu} - \frac{2}{M_A^2} \varepsilon_{\mu\nu\alpha\beta} \left(i\kappa_1^{AA} \langle \chi_- \rangle A^{\alpha\beta} + i\kappa_2^{AA} \{ \chi_-, A^{\alpha\beta} \} + \frac{1}{2} \kappa^{AAP} \{ P, A^{\alpha\beta} \} \right), \end{aligned}$$

we get for the $O(p^4)$ part of the Lagrangian

$$\begin{aligned} &\mathcal{L}_{RR,kin}^{(4)} + \mathcal{L}_R^{(4)} \rightarrow \mathcal{L}_{RR,kin}^{(4)} + \mathcal{L}_R^{(4)} \\ = &\mathcal{L}_{RR,kin}^{(4)} + \mathcal{L}_R^{(4)} \\ &- \kappa_1^{VV} \mathcal{O}_1^{VV} - \kappa_2^{VV} \mathcal{O}_2^{VV} - \kappa^{VVP} \mathcal{O}^{VVP} - \kappa_1^{AA} \mathcal{O}_1^{AA} - \kappa_2^{AA} \mathcal{O}_2^{AA} - \kappa^{AAP} \mathcal{O}^{AAP} \\ &- \frac{F_V}{\sqrt{2}M_V^2} \left(\kappa_1^{VV} \mathcal{O}_{13}^V + \kappa_2^{VV} \mathcal{O}_{14}^V + \frac{1}{2} \kappa^{VVP} \mathcal{O}_3^{PV} \right) \\ &- \frac{iG_V}{\sqrt{2}M_V^2} (2i\kappa_1^{VV} \mathcal{O}_{18}^V + 2i\kappa_2^{VV} \mathcal{O}_9^V - i\kappa^{VVP} \mathcal{O}_1^{PV}) \\ &- \frac{F_A}{\sqrt{2}M_A^2} \left(\kappa_1^{AA} \mathcal{O}_9^A + \kappa_2^{AA} \mathcal{O}_{11}^A + \frac{1}{2} \kappa^{AAP} \mathcal{O}_1^{PA} \right) + O(p^8). \end{aligned}$$

At the same time, the same redefinition applied to $\mathcal{L}_{R\chi T}^{(6, \text{odd})}$ generates only the additional terms of the order $O(p^8)$ and higher, which can be neglected as described above. We can thus eliminate the operators $\mathcal{O}_{1,2}^{VV}$, \mathcal{O}^{VVP} , $\mathcal{O}_{1,2}^{AA}$ and \mathcal{O}^{AAP} and include their influence on the $O(p^6)$ LECs effectively into the constants $\overline{\kappa}_{13}^V$, $\overline{\kappa}_{14}^V$, $\overline{\kappa}_3^{PV}$, $\overline{\kappa}_{18}^V$, $\overline{\kappa}_9^V$, $\overline{\kappa}_1^{PV}$ and $\overline{\kappa}_9^A$, $\overline{\kappa}_{11}^A$, $\overline{\kappa}_1^{PA}$.

B.2 Elimination of \mathcal{O}_i^{VA} and \mathcal{O}^{VAS}

In the same way we can eliminate also the mixed bilinear terms using the field redefinition

$$\begin{aligned} V_{\mu\nu} &\rightarrow V_{\mu\nu} - \frac{1}{M_V^2} \varepsilon_{\mu\nu\alpha\sigma} \left(i\kappa_1^{VA} g_\beta^\sigma [A^{\alpha\beta}, u^\rho u_\rho] + i\kappa_2^{VA} (A^{\alpha\beta} u_\beta u^\sigma - u^\sigma u_\beta A^{\alpha\beta}) \right. \\ &\quad + i\kappa_3^{VA} (A^{\alpha\beta} u^\sigma u_\beta - u_\beta u^\sigma A^{\alpha\beta}) + i\kappa_4^{VA} (u_\beta A^{\alpha\beta} u^\sigma - u^\sigma A^{\alpha\beta} u_\beta) \\ &\quad \left. + \kappa_5^{VA} \{ A^{\alpha\beta}, f_+^{\sigma\rho} \} g_{\beta\rho} + i\kappa_6^{VA} [A^{\alpha\beta}, \chi_+] g_\beta^\sigma + i\kappa^{VAS} [A^{\alpha\beta}, S] g_\beta^\sigma \right), \\ A_{\mu\nu} &\rightarrow A_{\mu\nu} - \frac{1}{M_A^2} \varepsilon_{\alpha\beta\mu\sigma} \left(i\kappa_1^{VA} g_\nu^\sigma [u^\rho u_\rho, V^{\alpha\beta}] + i\kappa_2^{VA} (u^\nu u^\sigma V^{\alpha\beta} - V^{\alpha\beta} u^\sigma u^\nu) \right. \\ &\quad + i\kappa_3^{VA} (u^\sigma u^\nu V^{\alpha\beta} - V^{\alpha\beta} u^\nu u^\sigma) + i\kappa_4^{VA} (u^\sigma V^{\alpha\beta} u^\nu - u^\nu V^{\alpha\beta} u^\sigma) \\ &\quad \left. + \kappa_5^{VA} \{ V^{\alpha\beta}, f_+^{\sigma\rho} \} g_{\rho\nu} + i\kappa_6^{VA} [\chi_+, V^{\alpha\beta}] g_\nu^\sigma + i\kappa^{VAS} [S, V^{\alpha\beta}] g_\nu^\sigma \right). \end{aligned}$$

We get then

$$\begin{aligned} \frac{1}{4}M_V^2\langle V^{\mu\nu}V_{\mu\nu}\rangle + \frac{1}{4}M_A^2\langle A^{\mu\nu}A_{\mu\nu}\rangle &\rightarrow \frac{1}{4}M_V^2\langle V^{\mu\nu}V_{\mu\nu}\rangle + \frac{1}{4}M_A^2\langle A^{\mu\nu}A_{\mu\nu}\rangle \\ &- \kappa_1^{VA}\mathcal{O}_1^{VA} - \kappa_2^{VA}\mathcal{O}_2^{VA} - \kappa_3^{VA}\mathcal{O}_3^{VA} - \kappa_4^{VA}\mathcal{O}_4^{VA} - \kappa_5^{VA}\mathcal{O}_5^{VA} - \kappa_6^{VA}\mathcal{O}_6^{VA} - \kappa^{VAS}\mathcal{O}^{VAS} \end{aligned}$$

and the operators \mathcal{O}_i^{VA} and \mathcal{O}^{VAS} are thus eliminated. The only relevant additional effect of the redefinition comes from transformation of $\mathcal{L}_R^{(4)}$

$$\begin{aligned} \frac{F_V}{2\sqrt{2}}\langle V_{\mu\nu}f_+^{\mu\nu}\rangle &\rightarrow \frac{F_V}{2\sqrt{2}}\langle V_{\mu\nu}f_+^{\mu\nu}\rangle - \frac{F_V}{2\sqrt{2}M_V^2}\left[-\kappa_1^{VA}\mathcal{O}_4^A + \kappa_2^{VA}(\mathcal{O}_6^A - \frac{1}{2}\mathcal{O}_4^A) \right. \\ &\quad \left. + \kappa_3^{VA}(\mathcal{O}_5^A - \frac{1}{2}\mathcal{O}_4^A) + \kappa_4^{VA}\mathcal{O}_7^A - \kappa_6^{VA}\mathcal{O}_{14}^A + \kappa^{VAS}\mathcal{O}_1^{SA}\right], \\ \frac{F_A}{2\sqrt{2}}\langle A_{\mu\nu}f_-^{\mu\nu}\rangle &\rightarrow \frac{F_A}{2\sqrt{2}}\langle A_{\mu\nu}f_-^{\mu\nu}\rangle - \frac{F_A}{2\sqrt{2}M_A^2}\left[\kappa_1^{VA}\mathcal{O}_5^V + \kappa_2^{VA}\mathcal{O}_8^V + \kappa_3^{VA}\mathcal{O}_6^V + \kappa_4^{VA}\mathcal{O}_7^V \right. \\ &\quad \left. - \kappa_5^{VA}\mathcal{O}_{11}^V + \kappa_6^{VA}\mathcal{O}_{15}^V - \kappa^{VAS}\mathcal{O}_1^{SV}\right], \\ \frac{iG_V}{2\sqrt{2}}\langle V_{\mu\nu}[u^\mu, u^\nu]\rangle &\rightarrow \frac{iG_V}{2\sqrt{2}}\langle V_{\mu\nu}[u^\mu, u^\nu]\rangle + \frac{G_V}{2\sqrt{2}M_V^2}\left[-2\kappa_1^{VA}\mathcal{O}_1^A + \kappa_2^{VA}(\mathcal{O}_2^A - \mathcal{O}_1^A) \right. \\ &\quad \left. + \kappa_3^{VA}(\mathcal{O}_2^A - \mathcal{O}_1^A) + \kappa_4^{VA}(\mathcal{O}_1^A - \mathcal{O}_2^A) \right. \\ &\quad \left. + \kappa_5^{VA}(\mathcal{O}_5^A - \mathcal{O}_6^A) + 2\kappa_6^{VA}\mathcal{O}_{13}^A + 2\kappa^{VAS}\mathcal{O}_2^{SA}\right]. \end{aligned}$$

Here we have used that

$$\langle\{A^{\alpha\beta}, f_+^{\sigma\rho}\}f_+^{\mu\nu}\rangle g_{\beta\rho}\varepsilon_{\mu\nu\alpha\sigma} = 0$$

and other similar consequences of the Shouten identity.

B.3 Elimination of \mathcal{O}_1^{PA} , \mathcal{O}_1^{SV} , \mathcal{O}_i^{PV} and \mathcal{O}_i^{SA}

Finally we can further eliminate another terms by the redefinitions

$$\begin{aligned} S &\rightarrow S + \frac{1}{2M_S^2}\varepsilon_{\mu\nu\alpha\beta}\left(i\kappa_1^{SA}[f_+^{\alpha\beta}, A^{\mu\nu}] + \kappa_2^{SA}[u^\alpha u^\beta, A^{\mu\nu}]\right), \\ P &\rightarrow P + \frac{1}{2M_P^2}\varepsilon_{\mu\nu\alpha\beta}\left(\kappa_1^{PA}\{A^{\mu\nu}, f_-^{\alpha\beta}\} + i\kappa_1^{PV}\{V^{\mu\nu}, u^\alpha u^\beta\} \right. \\ &\quad \left. + i\kappa_2^{PV}u^\beta V^{\mu\nu}u^\alpha + \kappa_3^{PV}\{V^{\mu\nu}, f_+^{\alpha\beta}\}\right), \\ A_{\mu\nu} &\rightarrow A_{\mu\nu} - \frac{1}{M_A^2}\varepsilon_{\mu\nu\alpha\beta}\left(i\kappa_1^{SA}[S, f_+^{\alpha\beta}] + \kappa_2^{SA}[S, u^\alpha u^\beta] + \kappa_1^{PA}\{P, f_-^{\alpha\beta}\}\right) \\ V_{\mu\nu} &\rightarrow V_{\mu\nu} - \frac{1}{M_V^2}\varepsilon_{\mu\nu\alpha\beta}\left(i\kappa_1^{PV}\{P, u^\alpha u^\beta\} + i\kappa_2^{PV}u^\alpha P u^\beta + \kappa_3^{PV}\{P, f_+^{\alpha\beta}\}\right). \end{aligned}$$

We get then

$$\begin{aligned} &-\frac{1}{2}M_P^2\langle PP\rangle - \frac{1}{2}M_S^2\langle SS\rangle + \frac{1}{4}M_A^2\langle A^{\mu\nu}A_{\mu\nu}\rangle + \frac{1}{4}M_V^2\langle V^{\mu\nu}V_{\mu\nu}\rangle \\ \rightarrow &-\frac{1}{2}M_P^2\langle PP\rangle - \frac{1}{2}M_S^2\langle SS\rangle + \frac{1}{4}M_A^2\langle A^{\mu\nu}A_{\mu\nu}\rangle + \frac{1}{4}M_V^2\langle V^{\mu\nu}V_{\mu\nu}\rangle \end{aligned}$$

$$-\kappa_1^{PA}\mathcal{O}_1^{PA}-\kappa_1^{PV}\mathcal{O}_1^{PV}-\kappa_2^{PV}\mathcal{O}_2^{PV}-\kappa_3^{PV}\mathcal{O}_3^{PV}-\kappa_1^{SA}\mathcal{O}_1^{SA}-\kappa_2^{SA}\mathcal{O}_2^{SA}-\kappa_1^{SV}\mathcal{O}_1^{SV},$$

therefore the operators \mathcal{O}_i^{SA} , \mathcal{O}_i^{PV} and \mathcal{O}_1^{PA} are eliminated. We get additional contributions

$$\begin{aligned} c_d\langle Su^\mu u_\mu\rangle &\rightarrow c_d\langle Su^\mu u_\mu\rangle + \frac{c_d}{2M_S^2}(-\kappa_1^{SA}\mathcal{O}_4^A - \kappa_2^{SA}\mathcal{O}_1^A - \kappa_1^{SV}\mathcal{O}_5^V) \\ c_m\langle S\chi_+\rangle &\rightarrow c_m\langle S\chi_+\rangle + \frac{c_m}{2M_S^2}(-\kappa_1^{SA}\mathcal{O}_{14}^A + \kappa_2^{SA}\mathcal{O}_{13}^A - \kappa_1^{SV}\mathcal{O}_{15}^V) \\ id_m\langle P\chi_-\rangle &\rightarrow id_m\langle P\chi_-\rangle + \frac{d_m}{2M_P^2}(\kappa_1^{PA}\mathcal{O}_{11}^A - \kappa_1^{PV}\mathcal{O}_9^V - \kappa_2^{PV}\mathcal{O}_{10}^V + \kappa_3^{PV}\mathcal{O}_{14}^V) \\ i\frac{d_{m0}}{N_F}\langle P\rangle\langle\chi_-\rangle &\rightarrow i\frac{d_{m0}}{N_F}\langle P\rangle\langle\chi_-\rangle + \frac{d_{m0}}{2N_F M_P^2}(2\kappa_1^{PA}\mathcal{O}_9^A - 2\kappa_1^{PV}\mathcal{O}_{18}^V - \kappa_2^{PV}\mathcal{O}_{18}^V + 2\kappa_3^{PV}\mathcal{O}_{13}^V) \\ \frac{F_A}{2\sqrt{2}}\langle A_{\mu\nu}f_-^{\mu\nu}\rangle &\rightarrow \frac{F_A}{2\sqrt{2}}\langle A_{\mu\nu}f_-^{\mu\nu}\rangle - \frac{F_A}{2\sqrt{2}M_A^2}(\kappa_1^{SA}\mathcal{O}_2^S - \kappa_2^{SA}\mathcal{O}_1^S + \kappa_1^{PA}\mathcal{O}_1^P) \\ \frac{F_V}{2\sqrt{2}}\langle V_{\mu\nu}f_+^{\mu\nu}\rangle &\rightarrow \frac{F_V}{2\sqrt{2}}\langle V_{\mu\nu}f_+^{\mu\nu}\rangle - \frac{F_V}{2\sqrt{2}M_V^2}(\kappa_1^{PV}\mathcal{O}_3^P - \kappa_2^{PV}\mathcal{O}_2^P + \kappa_3^{PV}\mathcal{O}_5^P - \kappa_1^{SV}\mathcal{O}_2^S) \\ \frac{iG_V}{2\sqrt{2}}\langle V_{\mu\nu}[u^\mu, u^\nu]\rangle &\rightarrow \frac{iG_V}{2\sqrt{2}}\langle V_{\mu\nu}[u^\mu, u^\nu]\rangle + \frac{G_V}{2\sqrt{2}M_V^2}(4\kappa_1^{PV}\mathcal{O}_4^P - 2\kappa_2^{PV}\mathcal{O}_4^P - 2\kappa_3^{PV}\mathcal{O}_3^P + 2\kappa_1^{SV}\mathcal{O}_1^S) \end{aligned}$$

B.4 The effective couplings $\overline{\kappa_i^X}$

Putting the result of previous subsections together we get the parameters $\overline{\kappa_i^X}$ of the reparameterized and truncated Lagrangian $\overline{\mathcal{L}}_{R\chi T}^{(6, \text{odd})}$, which is relevant for the saturation of ChPT LECs, as a functions of the parameters κ_i^X . As we have discussed above, the LECs have to depend on the couplings κ_i^X of the original Lagrangian $\mathcal{L}_{R\chi T}^{(6, \text{odd})}$ only through their particular combinations $\overline{\kappa_i^X}$. We have proved this by means of direct calculation as a nontrivial check of the formulae (91).

$$\begin{aligned} \overline{\kappa_1^V} &= \kappa_1^V \\ \overline{\kappa_2^V} &= \kappa_2^V \\ \overline{\kappa_3^V} &= \kappa_3^V \\ \overline{\kappa_4^V} &= \kappa_4^V \\ \overline{\kappa_5^V} &= \kappa_5^V - \frac{c_d}{2M_S^2}\left(\kappa_1^{SV} + \frac{F_A}{2\sqrt{2}M_A^2}\kappa^{VAS}\right) - \frac{F_A}{2\sqrt{2}M_A^2}\kappa_1^{VA} \\ \overline{\kappa_6^V} &= \kappa_6^V - \frac{F_A}{2\sqrt{2}M_A^2}\kappa_3^{VA} \\ \overline{\kappa_7^V} &= \kappa_7^V - \frac{F_A}{2\sqrt{2}M_A^2}\kappa_4^{VA} \\ \overline{\kappa_8^V} &= \kappa_8^V - \frac{F_A}{2\sqrt{2}M_A^2}\kappa_2^{VA} \\ \overline{\kappa_9^V} &= \kappa_9^V + \frac{2G_V\kappa_2^{VV}}{\sqrt{2}M_V^2} - \frac{d_m}{2M_P^2}\left(\kappa_1^{PV} - \frac{2G_V\kappa^{VVP}}{2\sqrt{2}M_V^2}\right) \end{aligned}$$

$$\begin{aligned}
\overline{\kappa_{10}^V} &= \kappa_{10}^V - \frac{d_m}{2M_P^2} \kappa_2^{PV} \\
\overline{\kappa_{11}^V} &= \kappa_{11}^V + \frac{F_A}{2\sqrt{2}M_A^2} \kappa_5^{VA} \\
\overline{\kappa_{12}^V} &= \kappa_{12}^V \\
\overline{\kappa_{13}^V} &= \kappa_{13}^V + \frac{d_{m0}}{N_F M_P^2} \left(\kappa_3^{PV} - \frac{F_V \kappa^{VVP}}{2\sqrt{2}M_V^2} \right) - \frac{F_V \kappa_1^{VV}}{\sqrt{2}M_V^2} \\
\overline{\kappa_{14}^V} &= \kappa_{14}^V - \frac{F_V \kappa_2^{VV}}{\sqrt{2}M_V^2} + \frac{d_m}{2M_P^2} \left(\kappa_3^{PV} - \frac{F_V \kappa^{VVP}}{2\sqrt{2}M_V^2} \right) \\
\overline{\kappa_{15}^V} &= \kappa_{15}^V - \frac{c_m}{2M_S^2} \left(\kappa_1^{SV} + \frac{F_A}{2\sqrt{2}M_A^2} \kappa^{VAS} \right) - \frac{F_A}{2\sqrt{2}M_A^2} \kappa_6^{VA} \\
\overline{\kappa_{16}^V} &= \kappa_{16}^V \\
\overline{\kappa_{17}^V} &= \kappa_{17}^V \\
\overline{\kappa_{18}^V} &= \kappa_{18}^V - \frac{d_{m0}}{2N_F M_P^2} \left(2 \left(\kappa_1^{PV} - \frac{2G_V \kappa^{VVP}}{2\sqrt{2}M_V^2} \right) + \kappa_2^{PV} \right) + \frac{2G_V \kappa_1^{VV}}{\sqrt{2}M_V^2} \\
\overline{\kappa_1^A} &= \kappa_1^A - \frac{c_d}{2M_S^2} \left(\kappa_2^{SA} + \frac{G_V}{\sqrt{2}M_V^2} \kappa^{VAS} \right) - \frac{G_V}{2\sqrt{2}M_V^2} (2\kappa_1^{VA} + \kappa_2^{VA} + \kappa_3^{VA} - \kappa_4^{VA}) \\
\overline{\kappa_2^A} &= \kappa_2^A + \frac{G_V}{2\sqrt{2}M_V^2} (\kappa_2^{VA} + \kappa_3^{VA} - \kappa_4^{VA}) \\
\overline{\kappa_3^A} &= \kappa_3^A \\
\overline{\kappa_4^A} &= \kappa_4^A - \frac{c_d}{2M_S^2} \left(\kappa_1^{SA} - \frac{F_V}{2\sqrt{2}M_V^2} \kappa^{VAS} \right) + \frac{F_V}{2\sqrt{2}M_V^2} \left(\kappa_1^{VA} + \frac{1}{2} \kappa_2^{VA} + \frac{1}{2} \kappa_3^{VA} \right) \\
\overline{\kappa_5^A} &= \kappa_5^A - \frac{F_V}{2\sqrt{2}M_V^2} \kappa_3^{VA} + \frac{G_V}{2\sqrt{2}M_V^2} \kappa_5^{VA} \\
\overline{\kappa_6^A} &= \kappa_6^A - \frac{F_V}{2\sqrt{2}M_V^2} \kappa_2^{VA} - \frac{G_V}{2\sqrt{2}M_V^2} \kappa_5^{VA} \\
\overline{\kappa_7^A} &= \kappa_7^A - \frac{F_V}{2\sqrt{2}M_V^2} \kappa_4^{VA} \\
\overline{\kappa_8^A} &= \kappa_8^A \\
\overline{\kappa_9^A} &= \kappa_9^A + \frac{d_{m0}}{N_F M_P^2} \left(\kappa_1^{PA} - \frac{F_A \kappa^{AAP}}{2\sqrt{2}M_A^2} \right) - \frac{F_A \kappa_1^{AA}}{\sqrt{2}M_A^2} \\
\overline{\kappa_{10}^A} &= \kappa_{10}^A \\
\overline{\kappa_{11}^A} &= \kappa_{11}^A - \frac{F_A \kappa_2^{AA}}{\sqrt{2}M_A^2} + \frac{d_m}{2M_P^2} \left(\kappa_1^{PA} - \frac{F_A \kappa^{AAP}}{2\sqrt{2}M_A^2} \right) \\
\overline{\kappa_{12}^A} &= \kappa_{12}^A
\end{aligned}$$

$$\begin{aligned}
\overline{\kappa_{13}^A} &= \kappa_{13}^A + \frac{G_V}{\sqrt{2}M_V^2}\kappa_6^{VA} + \frac{c_m}{2M_S^2}\left(\kappa_2^{SA} + \frac{G_V}{\sqrt{2}M_V^2}\kappa^{VAS}\right) \\
\overline{\kappa_{14}^A} &= \kappa_{14}^A + \frac{F_V}{2\sqrt{2}M_V^2}\kappa_6^{VA} - \frac{c_m}{2M_S^2}\left(\kappa_1^{SA} - \frac{F_V}{2\sqrt{2}M_V^2}\kappa^{VAS}\right) \\
\overline{\kappa_{15}^A} &= \kappa_{15}^A \\
\overline{\kappa_{16}^A} &= \kappa_{16}^A \\
\overline{\kappa_1^S} &= \kappa_1^S + \frac{F_A}{2\sqrt{2}M_A^2}\left(\kappa_2^{SA} + \frac{G_V}{\sqrt{2}M_V^2}\kappa^{VAS}\right) + \frac{G_V}{\sqrt{2}M_V^2}\left(\kappa_1^{SV} + \frac{F_A}{2\sqrt{2}M_A^2}\kappa^{VAS}\right) \\
\overline{\kappa_2^S} &= \kappa_2^S - \frac{F_A}{2\sqrt{2}M_A^2}\left(\kappa_1^{SA} - \frac{F_V}{2\sqrt{2}M_V^2}\kappa^{VAS}\right) + \frac{F_V}{2\sqrt{2}M_V^2}\left(\kappa_1^{SV} + \frac{F_A}{2\sqrt{2}M_A^2}\kappa^{VAS}\right) \\
\overline{\kappa_1^P} &= \kappa_1^P - \frac{F_A}{2\sqrt{2}M_A^2}\left(\kappa_1^{PA} - \frac{F_A\kappa^{AAP}}{2\sqrt{2}M_A^2}\right) \\
\overline{\kappa_2^P} &= \kappa_2^P + \frac{F_V}{2\sqrt{2}M_V^2}\kappa_2^{PV} \\
\overline{\kappa_3^P} &= \kappa_3^P - \frac{F_V}{2\sqrt{2}M_V^2}\left(\kappa_1^{PV} - \frac{2G_V\kappa^{VVP}}{2\sqrt{2}M_V^2}\right) - \frac{G_V}{\sqrt{2}M_V^2}\left(\kappa_3^{PV} - \frac{F_V\kappa^{VVP}}{2\sqrt{2}M_V^2}\right) \\
\overline{\kappa_4^P} &= \kappa_4^P + \frac{G_V}{\sqrt{2}M_V^2}\left(2\left(\kappa_1^{PV} - \frac{2G_V\kappa^{VVP}}{2\sqrt{2}M_V^2}\right) - \kappa_2^{PV}\right) \\
\overline{\kappa_5^P} &= \kappa_5^P - \frac{F_V}{2\sqrt{2}M_V^2}\left(\kappa_3^{PV} - \frac{F_V\kappa^{VVP}}{2\sqrt{2}M_V^2}\right) \\
\overline{\kappa_3^{VV}} &= \kappa_3^{VV} \\
\overline{\kappa_4^{VV}} &= \kappa_4^{VV} \\
\overline{\kappa_2^{SV}} &= \kappa_2^{SV} \\
\overline{\kappa_2^{PA}} &= \kappa_2^{PA}
\end{aligned}$$

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